

One-Loop Corrections to Two-Quark Three-Gluon Amplitudes

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Abstract

We present the one-loop QCD amplitudes for two external massless quarks and three external gluons ($\bar{q}qggg$). This completes the set of one-loop amplitudes needed for the next-to-leading-order corrections to three-jet production at hadron colliders. We also discuss how to use group theory and supersymmetry to minimize the amount of calculation required for the more general case of one-loop two-quark n -gluon amplitudes. We use collinear limits to provide a stringent check on the amplitudes.

Submitted to Nuclear Physics B

[#]Research supported in part by the US Department of Energy under grant DE-FG03-91ER40662 and in part by the Alfred P. Sloan Foundation under grant BR-3222.

^{*}Research supported by the US Department of Energy under grant DE-AC03-76SF00515.

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1. Introduction

Jet physics at hadron colliders allows one to confront the theoretical predictions of QCD with experimental results and thereby probe for new physics at the highest possible energies. Yet precise comparisons between theory and experiment are hampered by the lack of calculations beyond the leading order of perturbation theory, for all but the simplest processes. In pure QCD, the next-to-leading-order corrections computed to date [1] have relied on the one-loop amplitudes for four external partons, first calculated by Ellis and Sexton [2]. More recently, we have calculated the one-loop amplitudes for five external gluons ($ggggg$) [3], and Kunszt, Signer, and Trócsányi (KST) have calculated the amplitudes for four quarks and a gluon ($\bar{q}q\bar{q}qg$) [4]. In this paper we present the remaining one-loop five-parton amplitudes, for two (massless) quarks and three gluons ($\bar{q}qggg$). Combining these analytic results with the known six-parton tree amplitudes [5,6], one can now construct numerical programs for next-to-leading-order corrections to three-jet production at hadron colliders, and examine the structure of jets, for example dependence of cross-sections on the cone size, beyond the leading non-trivial order probed in next-to-leading order two-jet programs [1,7]. Computation of the ratio of three-jet to two-jet events at hadron colliders at next-to-leading order in α_s would also make possible the measurement of α_s in purely hadronic processes and at the largest energy scales available.

Many methods developed in recent years can be used to simplify the computation of one-loop multi-parton amplitudes, including spinor helicity methods [8], color decomposition of amplitudes [5,9,10], string-based techniques [11,12,13,14,3], supersymmetry Ward identities [15,16], supersymmetry-based decompositions [3,17,18], and perturbative unitarity [19,20,21]; all of these techniques have been used to obtain the amplitudes presented in this paper.

We have found it useful to organize the calculation in terms of gauge-invariant, color-ordered building blocks, dubbed *primitive amplitudes*. We show in the next section that all of the kinematic coefficients (partial amplitudes) appearing in the color decomposition of amplitudes with two quarks and $(n-2)$ gluons can be expressed as sums over permutations of gauge invariant primitive amplitudes. The analytic structure of a primitive amplitude is generally simpler than that of a partial amplitude; a primitive amplitude receives contributions only from diagrams with a fixed ordering of external legs, while the generic partial amplitude receives contributions from multiple orderings. Thus, fewer kinematic invariants appear in each primitive amplitude. Although this organization was motivated in part by string theory, our discussion is entirely field-theoretic.

We use supersymmetry to reduce the number of quantities to be calculated. QCD amplitudes may be decomposed in terms of supersymmetric and non-supersymmetric parts. Through use of supersymmetry Ward identities, the supersymmetric parts of amplitudes with two external quarks and three external gluons may be obtained directly from the previously calculated five-gluon amplitudes [3].

We have also made use of the cut-reconstruction method described in refs. [20,21]. If certain power-counting criteria are satisfied, amplitudes are entirely constructible from their cuts. Although QCD amplitudes are generally not cut-constructible, by taking linear combinations of QCD amplitudes with ones involving scalars and/or gluinos, the QCD amplitudes may be separated into cut-constructible and non-cut-constructible parts. We have used this unitarity-based technique to obtain the cut-constructible components of some of the primitive amplitudes for $\bar{q}qggg$ (those that enter into the subleading-in-color contributions to the virtual part of the cross-section). Here the cut-constructible components are formed by adding to the desired diagrams a new set of diagrams, which differs only in the replacement of virtual gluons in the loop by scalars. For a specific choice of the Yukawa coupling between the scalars and the quark line, the sum of gluon and scalar diagrams satisfies the power-counting criteria (see ref. [21]). We then calculate the scalar contributions directly; they are not cut-constructible, but they are easier to calculate directly than the full gluon contributions. Finally we reassemble the desired gluon contributions.

In order to ensure the correctness of the amplitudes, we have performed a number of checks. As the momenta of two external legs become collinear the amplitudes must factorize properly. We have verified this factorization for all amplitudes in all channels. This provides an extremely stringent constraint on the amplitudes. In fact, this constraint is sufficiently powerful that it has been used to construct ansätze for a number of amplitudes with fixed helicities but an arbitrary number of external legs [22,23,20], which were then proven correct by either recursive [24] or unitarity [20,21] techniques. (The recursive and unitarity techniques have also been used to construct a variety of other one-loop amplitudes with an arbitrary number of external legs [24,20,21].)

We have performed additional checks on certain helicity amplitudes by computing all diagrams that enter into a supersymmetry Ward identity [15], and explicitly verifying the identity. Not only does this provide a check on amplitudes presented in this paper, but also on the supersymmetric combinations of the five-gluon amplitudes presented in ref. [3]. (A similar supersymmetry check using the five-gluon amplitudes has been carried out [25] for the $\bar{q}q\bar{q}qg$ amplitudes reported in ref. [4].) As a final check, we have verified that the cuts in some amplitudes obtained by more direct diagrammatic means are consistent with unitarity.

In section 2, we give the $SU(N_c)$ color decomposition for amplitudes involving two external

quarks and $n - 2$ external gluons, as a sum of color factors multiplied by partial amplitudes. We also give a formula for the sum over colors of the interference between tree and one-loop $\bar{q}qggg$ amplitudes, in terms of partial amplitudes; this formula is required for the virtual part of the color-summed parton-level cross-section. The primitive amplitudes, which form the gauge-invariant building blocks for the amplitudes, are described in section 3. The precise relation of the primitive amplitudes to the partial amplitudes is given in section 4. In section 5 we give the main results of the paper, the primitive amplitudes for $\bar{q}qggg$. Section 6 contains our conclusions. Four appendices contain technical details related to color algebra and collinear checks. Appendix I provides a derivation of the relation between primitive and partial amplitudes. Appendix II collects the one-loop four-point amplitudes [16,11] that appear in collinear limits of $\bar{q}qggg$ amplitudes, namely $gggg$ and $\bar{q}qgg$. Appendix III then illustrates the procedure for carrying out collinear checks, using these amplitudes and “splitting amplitudes” from ref. [20]. Finally, appendix IV shows how to use the two-quark $(n - 2)$ -gluon primitive amplitudes to construct amplitudes where some of the gluons are replaced by photons.

2. Color Decomposition for Two-Quark $(n - 2)$ -Gluon Amplitudes

In this section we describe a color decomposition of the one-loop two-quark $(n - 2)$ -gluon amplitude $\bar{q}qg \dots g$, in terms of group-theoretic factors (color structures) multiplied by kinematic functions called *partial amplitudes*. In the following sections, we shall give formulae for all of the partial amplitudes in terms of color-ordered, gauge-invariant building blocks called *primitive amplitudes*. A primitive amplitude is defined as the sum of all one-loop diagrams in which the n external legs have a fixed order around the loop (the color order), with some additional restrictions to be described in the following section.

For the $\bar{q}qg \dots g$ amplitudes, let particle 1 be an antiquark, transforming in the \overline{N}_c representation of $SU(N_c)$, with color index \bar{i}_1 , and let particle 2 be a quark, transforming in the N_c with index i_2 . Denote these particles by $1_{\bar{q}}$ and 2_q in order to distinguish them from the remaining gluons, particles 3 to n , transforming in the adjoint representation with indices a_3, \dots, a_n . We also allow for n_f Weyl fermions and n_s complex scalars circulating in the loop, both in the $(N_c + \overline{N}_c)$ representation (n_f flavors of massless quarks and n_s massless scalars).

The general strategy for obtaining multi-parton color decompositions is to rewrite the $SU(N_c)$ structure constants f^{abc} in terms of the group generators in the fundamental representation T^a , normalized so that $\text{Tr}(T^a T^b) = \delta^{ab}$,

$$f^{abc} = -\frac{i}{\sqrt{2}} \text{Tr}([T^a, T^b] T^c). \quad (2.1)$$

Then one applies the $SU(N_c)$ Fierz identity

$$(X_1 T^a X_2) (Y_1 T^a Y_2) = (X_1 Y_2) (Y_1 X_2) - \frac{1}{N_c} (X_1 X_2) (Y_1 Y_2) , \quad (2.2)$$

where X_i, Y_i are strings of generator matrices T^{a_i} , in order to remove contracted color indices. In discussing amplitudes where all external particles are in the adjoint representation (such as amplitudes in supersymmetric QCD with no matter content), or trees consisting of only particles in the adjoint representation, one may (and should) replace the $SU(N_c)$ Fierz identity with the corresponding $U(N_c)$ identity, since it is simpler: the ‘photonic’ term decouples [5,9,10]. For the two-quark amplitudes under consideration, at tree level only one string survives and the color decomposition is [5]

$$\mathcal{A}_n^{\text{tree}}(1_{\bar{q}}, 2_q, 3, \dots, n) = g^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \dots T^{a_{\sigma(n)}})_{i_2}^{\bar{i}_1} A_n^{\text{tree}}(1_{\bar{q}}, 2_q; \sigma(3), \dots, \sigma(n)) , \quad (2.3)$$

where S_{n-2} is the permutation group on $n-2$ elements, and A_n^{tree} are the tree-level partial amplitudes. They are identical to the tree-level partial amplitudes for the process with gluinos replacing quarks, and are thereby related to the tree-level all-gluon amplitudes by supersymmetry Ward identities [15]. We adopt throughout the convention that all momenta are taken to be outgoing. Because the color indices have been stripped off from the partial amplitudes, there is no need to distinguish a quark leg from an anti-quark leg; charge conjugation relates the two choices.

At one loop an additional trace may survive, and the color decomposition is

$$\mathcal{A}_n(1_{\bar{q}}, 2_q, 3, \dots, n) = g^n \sum_{j=1}^{n-1} \sum_{\sigma \in S_{n-2}/S_{n;j}} \text{Gr}_{n;j}^{(\bar{q}q)}(\sigma(3), \dots, n) A_{n;j}(1_{\bar{q}}, 2_q; \sigma(3), \dots, n) , \quad (2.4)$$

where the color structures $\text{Gr}_{n;j}^{(\bar{q}q)}$ are defined by

$$\begin{aligned} \text{Gr}_{n;1}^{(\bar{q}q)}(3, \dots, n) &= N_c (T^{a_3} \dots T^{a_n})_{i_2}^{\bar{i}_1} , \\ \text{Gr}_{n;2}^{(\bar{q}q)}(3, 4, \dots, n) &= 0 , \\ \text{Gr}_{n;j}^{(\bar{q}q)}(3, \dots, j+1; j+2, \dots, n) &= \text{Tr}(T^{a_3} \dots T^{a_{j+1}}) (T^{a_{j+2}} \dots T^{a_n})_{i_2}^{\bar{i}_1} , \quad j = 3, \dots, n-2, \\ \text{Gr}_{n;n-1}^{(\bar{q}q)}(3, \dots, n) &= \text{Tr}(T^{a_3} \dots T^{a_n}) \delta_{i_2}^{\bar{i}_1} , \end{aligned} \quad (2.5)$$

and $S_{n;j} = Z_{j-1}$ is the subgroup of S_{n-2} that leaves $\text{Gr}_{n;j}^{(\bar{q}q)}$ invariant. When the permutation σ acts on a list of indices, it is to be applied to each index separately: $\sigma(3, \dots, n) \equiv \sigma(3), \dots, \sigma(n)$, etc. We refer to $A_{n;1}$ as the leading-color partial amplitude, and to the $A_{n;j>1}$ as subleading, because for large N_c , $A_{n;1}$ alone gives the leading contribution to the color-summed correction to the cross-section, obtained by interfering $\mathcal{A}_n^{\text{tree}}$ with \mathcal{A}_n . The explicit N_c in the definition of the

leading color structure $\text{Gr}_{n;1}^{(\bar{q}q)}$ — which is otherwise identical to the tree color structure — ensures that $A_{n;1}$ is $\mathcal{O}(1)$ for large N_c .

It is useful to recall the analogous color decomposition for n external particles in the adjoint representation, in particular for the pure super-Yang-Mills amplitude for two external gluinos and $n - 2$ gluons (similar to the decomposition for n -gluon amplitudes [10]),

$$\mathcal{A}_n^{\text{SUSY}}(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n) = g^n \sum_{j=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n / \tilde{S}_{n;j}} \text{Gr}_{n;j}(\sigma(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n)) A_{n;j}^{\text{SUSY}}(\sigma(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n)) , \quad (2.6)$$

where

$$\begin{aligned} \text{Gr}_{n;1}(1, 2, \dots, n) &= N_c \text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n}) , \\ \text{Gr}_{n;j}(1, \dots, j-1; j, \dots, n) &= \text{Tr}(T^{a_1} \dots T^{a_{j-1}}) \text{Tr}(T^{a_j} \dots T^{a_n}) , \quad j = 2, \dots, \lfloor n/2 \rfloor + 1, \end{aligned} \quad (2.7)$$

$\lfloor x \rfloor$ is the largest integer less than or equal to x , \tilde{g} stands for the gluino legs, and $\tilde{S}_{n;j}$ is the subset of S_n that leaves $\text{Gr}_{n;j}$ invariant. The similar structure of the gluino amplitudes and quark amplitudes will help in understanding how to simplify the organization of the latter, particularly the subleading-color contributions, $A_{n;j>1}$.

The partial amplitudes $A_{n;j}^{\text{SUSY}}$ for $\tilde{g}\tilde{g}ggg$ can be obtained from five-gluon results [3] and supersymmetry Ward identities [15]; these can in turn be used to reduce the work required in the quark case. Throughout this paper, we consider only supersymmetric amplitudes with no matter content. The use of supersymmetry in loop amplitudes implies a calculation using a supersymmetry-preserving regulator, such as dimensional reduction [26] or the four-dimensional helicity scheme [11] (which are very closely related at one loop). To obtain results for the quark amplitudes in other schemes, such as the conventional dimensional regularization method [27] (often called the ‘ $\overline{\text{MS}}$ ’ scheme in the literature), one must shift the partial amplitudes presented here by a quantity proportional to the tree amplitude; the constant of proportionality has been determined by Kunszt, Signer, and Trócsányi [16] (see eqn. (5.5)).

Unlike the all-external-gluon case, we can subdivide the two-gluino $(n - 2)$ -gluon color structures further, depending on the trace to which the fermion charge matrices belong. Define $\text{Gr}_{n;j}^s$ and $\text{Gr}_{n;j}^d$ to be respectively $\text{Gr}_{n;j}$ with fermion charge matrices in the same and different traces; furthermore, require for $\text{Gr}_{n;j}^s$ that the fermion charge matrices lie in the second trace, which

increases the number of such structures from $\lfloor n/2 \rfloor + 1$ to $n - 1$. We may then write

$$\begin{aligned} \mathcal{A}_n^{\text{SUSY}}(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n) &= g^n \sum_{j=1}^{n-1} \sum_{\sigma \in \Sigma_{n;j}^s} \text{Gr}_{n;j}^s(\sigma(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n)) A_{n;j}^{\text{SUSY}}(\sigma(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n)) \\ &\quad + g^n \sum_{j=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in \Sigma_{n;j}^d} \text{Gr}_{n;j}^d(\sigma(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n)) A_{n;j}^{\text{SUSY}}(\sigma(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n)) , \end{aligned} \quad (2.8)$$

where $\Sigma_{n;j}^s$ and $\Sigma_{n;j}^d$ are the sets of permutations of n elements that act nontrivially on $\text{Gr}_{n;j}^s$ and $\text{Gr}_{n;j}^d$ respectively, and preserve the assignment of fermion charge matrices to the two traces. In appendix I, we will find this separation useful when relating adjoint representation partial amplitudes to fundamental representation partial amplitudes.

Returning to the quark case, the explicit decomposition of the $\bar{q}qgg$ one-loop amplitude is

$$\begin{aligned} \mathcal{A}_4(1_{\bar{q}}, 2_q, 3, 4) &= g^4 \left[N_c (T^{a_3} T^{a_4})_{i_2}^{\bar{i}_1} A_{4;1}(1_{\bar{q}}, 2_q; 3, 4) + N_c (T^{a_4} T^{a_3})_{i_2}^{\bar{i}_1} A_{4;1}(1_{\bar{q}}, 2_q; 4, 3) \right. \\ &\quad \left. + \text{Tr}(T^{a_3} T^{a_4}) \delta_{i_2}^{\bar{i}_1} A_{4;3}(1_{\bar{q}}, 2_q; 3, 4) \right] , \end{aligned} \quad (2.9)$$

in agreement with the decomposition used by KST [16] (apart from notational differences such as the explicit factor of N_c in $\text{Gr}_{4;1}^{(\bar{q}q)}$ and the ordering of external legs).

The decomposition of the $\bar{q}qqgg$ amplitude is [28]

$$\begin{aligned} \mathcal{A}_5(1_{\bar{q}}, 2_q, 3, 4, 5) &= g^5 \left[N_c \sum_{\sigma \in S_3} (T^{a_{\sigma(3)}} T^{a_{\sigma(4)}} T^{a_{\sigma(5)}})_{i_2}^{\bar{i}_1} A_{5;1}(1_{\bar{q}}, 2_q; \sigma(3), \sigma(4), \sigma(5)) \right. \\ &\quad + \sum_{\sigma \in Z_3} \text{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) (T^{a_{\sigma(5)}})_{i_2}^{\bar{i}_1} A_{5;3}(1_{\bar{q}}, 2_q; \sigma(3), \sigma(4); \sigma(5)) \\ &\quad \left. + \text{Tr}(T^{a_3} T^{a_4} T^{a_5}) \delta_{i_2}^{\bar{i}_1} A_{5;4}(1_{\bar{q}}, 2_q; 3, 4, 5) + \text{Tr}(T^{a_5} T^{a_4} T^{a_3}) \delta_{i_2}^{\bar{i}_1} A_{5;4}(1_{\bar{q}}, 2_q; 5, 4, 3) \right] . \end{aligned} \quad (2.10)$$

In the partial amplitude $A_{5;3}$ an additional semicolon separates the gluon sandwiched between the quark indices (the last gluon in $A_{5;3}$) from the other two gluons.

The virtual part of the next-to-leading order correction to the parton-level cross-section is given by the sum over colors of the interference between the tree amplitude $\mathcal{A}_n^{\text{tree}}$ and the one-loop amplitude \mathcal{A}_n . Using the color decompositions (2.3) and (2.4), and the Fierz rules (2.2) it is straightforward to evaluate this color-sum in terms of partial amplitudes. Here we give the formula

for the five-point case, $\bar{q}qggg$,

$$\begin{aligned}
2 \operatorname{Re} \sum_{\text{colors}} [\mathcal{A}_5^{\text{tree}} * \mathcal{A}_5] &= 2g^8 \frac{N_c^2 - 1}{N_c} \operatorname{Re} \sum_{\sigma \in S_3} A_5^{\text{tree}*}(1_{\bar{q}}, 2_q; \sigma(3), \sigma(4), \sigma(5)) \\
&\times \left[(N_c^2 - 1)^2 A_{5;1}(1_{\bar{q}}, 2_q; \sigma(3), \sigma(4), \sigma(5)) \right. \\
&\quad - (N_c^2 - 1) \left(A_{5;1}(1_{\bar{q}}, 2_q; \sigma(4), \sigma(3), \sigma(5)) + A_{5;1}(1_{\bar{q}}, 2_q; \sigma(3), \sigma(5), \sigma(4)) \right. \\
&\quad \quad \left. - A_{5;3}(1_{\bar{q}}, 2_q; \sigma(4), \sigma(5); \sigma(3)) - A_{5;3}(1_{\bar{q}}, 2_q; \sigma(3), \sigma(4); \sigma(5)) \right) \\
&\quad + (N_c^2 + 1) A_{5;1}(1_{\bar{q}}, 2_q; \sigma(5), \sigma(4), \sigma(3)) + (N_c^2 - 2) A_{5;4}(1_{\bar{q}}, 2_q; \sigma(3), \sigma(4), \sigma(5)) \\
&\quad + A_{5;1}(1_{\bar{q}}, 2_q; \sigma(4), \sigma(5), \sigma(3)) + A_{5;1}(1_{\bar{q}}, 2_q; \sigma(5), \sigma(3), \sigma(4)) \\
&\quad \left. - A_{5;3}(1_{\bar{q}}, 2_q; \sigma(5), \sigma(3); \sigma(4)) - 2 A_{5;4}(1_{\bar{q}}, 2_q; \sigma(5), \sigma(4), \sigma(3)) \right], \tag{2.11}
\end{aligned}$$

where the sum is over all permutations of the three gluons.

3. Primitive Amplitudes

In this section we introduce a set of gauge-invariant, color-ordered building blocks, which we call *primitive amplitudes*, that suffice to determine all the two-quark $(n - 2)$ -gluon partial amplitudes $A_{n;j}$. Explicit expressions for primitive amplitudes tend to be much more compact than those for the generic partial amplitude, because the legs are ordered. Only a subset of the kinematic invariants $(\sum k_i)^2$ appear as arguments of logarithms or dilogarithms in a color-ordered set of diagrams, namely those where all the momenta are adjacent with respect to the ordering, and this leads to simpler analytic structure for such objects.

It is not obvious *a priori* that every partial amplitude can be expressed in terms of primitive amplitudes, because the generic partial amplitude receives contributions from diagrams with several *different* cyclic orderings of the external legs, and it does *not* receive contributions from certain classes of diagrams present in the primitive amplitudes. Nevertheless, it is possible to write every two-quark $(n - 2)$ -gluon partial amplitude as a sum over permutations of primitive amplitudes; one can show (using string-inspired groupings of diagrams) that the unwanted diagrams cancel out in the sum. We perform the necessary $SU(N_c)$ group theory manipulations in the double-line formalism [29]. The color decomposition for one-loop $\bar{q}qg \dots g$ amplitudes is analogous to that presented in ref. [10] for one-loop amplitudes with n external gluons, and the manipulations used to simplify subleading color contributions are similar to ones used in ref. [20]. We also employ supersymmetry [15,17,16], in order to reduce the number of primitive amplitudes that have to be calculated directly for $\bar{q}qggg$.

To present the primitive amplitudes, we will find it convenient to use the language of color-ordered diagrams. These are basically Feynman diagrams from which color indices have been stripped, the relevant signs encoded in the topology of the graph. The vertices are no longer symmetric under exchanges of legs, and thus the external ordering of the legs becomes important: two diagrams which would be equivalent as Feynman diagrams are generally no longer so as color-ordered diagrams. The reader may find a description of tree-level color-ordered Feynman rules in refs. [6,17]. One obtains them by using the trace representation (2.1) of the structure constants f^{abc} , as well as eqn. (2.2) or its $U(N_c)$ counterpart for adjoint states. One then dresses the Feynman diagrams using the doubled color line notation for the adjoint-representation propagators, and single color lines for fundamental-representation ones [29]. The two terms in equation (2.1), and in equation (2.2), mean that one Feynman diagram can generate many different color-flow diagrams. By convention, we draw diagrams so that the ordering of legs is clockwise around the loop.

In computing the coefficient of a particular color structure, or configuration of generator matrices T^{a_i} , one can then remove group theory factors from the vertices, arriving at the color-ordered rules shown in fig. 1, where the gluon Lorentz indices are μ, ν, ρ, λ and outgoing momenta are k, p, q , and the fermion \tilde{g} is taken to be in the adjoint representation. The cyclic ordering of the legs in fig. 1 is important since there is a relative sign between the two orderings of the three point vertices; if one interchanges any two of the legs the vertex changes by a sign. This sign follows from the antisymmetry of the structure constants f^{abc} .

$$\begin{aligned}
\begin{array}{c} \rho \\ q \\ \text{wavy} \\ \nu \\ p \end{array} \begin{array}{c} k \\ \mu \end{array} &= \frac{i}{\sqrt{2}} (\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho) \\
\begin{array}{c} \mu \\ \text{wavy} \\ \lambda \end{array} \begin{array}{c} \nu \\ \text{wavy} \\ \rho \end{array} &= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}) \\
\begin{array}{c} \mu \\ \text{wavy} \end{array} \begin{array}{c} \text{fermion} \end{array} &= \frac{i}{\sqrt{2}}\gamma_\mu & \begin{array}{c} \mu \\ \text{wavy} \end{array} \begin{array}{c} \nu \\ \text{wavy} \end{array} &= -i\frac{\eta_{\mu\nu}}{p^2} \\
\begin{array}{c} \mu \\ \text{wavy} \end{array} \begin{array}{c} \text{fermion} \end{array} &= -\frac{i}{\sqrt{2}}\gamma_\mu & \begin{array}{c} \text{fermion} \end{array} &= \frac{i}{\not{p}}
\end{aligned}$$

Figure 1. The color-ordered Feynman rules have antisymmetric three-vertices. Straight lines represent fermions, and wavy lines gluons.

In various steps of our explicit calculations it is advantageous [13,17,18] to use a different gauge than the standard Feynman gauge used in fig. 1, such as background-field gauge [30] and Gervais-Neveu gauge [31]. In the various gauges the color-ordered three- and four-gluon vertices look different. However, partial amplitudes and primitive amplitudes are gauge invariant (we prove the invariance of the latter at the end of this section); therefore the formulae below, expressing partial amplitudes as sums of primitive amplitudes, which are derived using the rules in fig. 1, will hold in any gauge.

If the fermion is in the fundamental representation, the same rules hold, but one of the three color lines at the vertex, the one flowing along side the fermion line, should be removed or ‘stripped off’. This procedure of ‘color-stripping’ equates a contribution to an amplitude with external fundamental-representation fermions to a contribution to an amplitude with external adjoint-representation fermions. The latter differ from the former in a way we shall make precise in appendix I.

In order to motivate the precise definition of primitive amplitudes for two quarks and $(n - 2)$ gluons, we first discuss amplitudes with external gluons only. Consider the set of diagrams contributing to the leading all-gluon partial amplitude $A_{n;1}(1, 2, \dots, n)$. The only diagrams that contribute are those with a single ordering $1 \dots n$ of legs around the loop, the ordering matching the associated color structure $\text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n})$, as depicted in fig. 2. The color-ordered diagrams for this partial amplitude may be further distinguished by whether a gluon, a fermion, or a scalar circulates around the loop; each contribution is separately gauge-invariant because the coefficients of n_f and n_s in the full amplitude must be gauge-invariant. We denote them by $A_{n;1}^{[J]}$, with J the spin of the circulating particle, $J = 1, \frac{1}{2}, 0$.

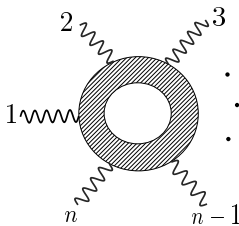


Figure 2. The external legs of color-ordered diagrams have a fixed clockwise ordering.

The subleading n -gluon partial amplitudes $A_{n;j>1}$ (which are only present when an adjoint particle, not a fundamental, circulates in the loop) generically receive contributions from diagrams with many different cyclic orderings. However, it is possible [20] to express the $A_{n;j}^{[J]}$ as a sum over permutations of the $A_{n;1}^{[J]}$, as we review in appendix I. Therefore the partial amplitudes $A_{n;1}^{[J]}$

suffice to construct the full n -gluon amplitude. On the other hand, other than separating the contributions of internal particles of different spin, it is not possible to find gauge-invariant subsets of the diagrams that contribute to $A_{n;1}^{[J]}$. When all external legs are gluons, then, the $A_{n;1}^{[J]}$ are the irreducible gauge-invariant pieces of the amplitude, and serve as primitive amplitudes.

If we were interested only in the two-gluino $(n-2)$ -gluon (supersymmetric) amplitude we could take the primitive amplitudes simply to be the leading-color partial amplitudes $A_{n;1}$, just as in the n -gluon case, because the same subleading-from-leading permutation formula that holds in the n -gluon case applies to mixed gluino-gluon amplitudes as well. However, for the $\bar{q}qg\dots g$ amplitudes, one has to divide the sets of color-ordered diagrams into finer pieces before such an approach can succeed. The need for a finer division of the diagrams arises from the color flow which goes solely in one direction along a quark line, but in both directions along a gluino line. The gluino amplitudes $A_{n;1}$ can be decomposed further in a gauge-invariant way; the pieces of this decomposition are the primitive amplitudes, which are also the pieces out of which we may form the amplitude for external quarks.

To ferret out these gauge-invariant subparts of the supersymmetric amplitude, we must trace the external fermion line through the diagram. Since the fermion line has an arrow, we can distinguish one-loop diagrams according to which side of the loop the fermion line is on. We define a ‘left’ class of diagrams to be those where, following the arrow, the fermion line passes to the left of the loop; the remaining diagrams are in the ‘right’ class. For example, in fig. 3a the fermion line enters the loop and turns left, passing to the left of the loop, so this is a ‘left’ diagram; fig. 3b is a ‘right’ diagram. In fig. 4a the external fermion line does not actually enter the loop, but still it passes to the left, so this is a ‘left’ diagram; fig. 4b is a ‘right’ diagram. In diagrams of the type shown in fig. 4 the particle circulating in the closed loop may be either a gluon, a fermion or a scalar; the left/right designation is applied in the same way. (The external fermion line is always used to make the distinction, not the fermion that might be in the closed loop.)

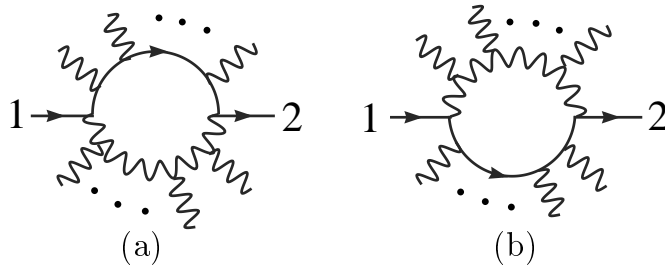


Figure 3. In diagram (a) the fermion line (following the arrow) turns ‘left’ on entering the loop, and in diagram (b) it turns ‘right’.

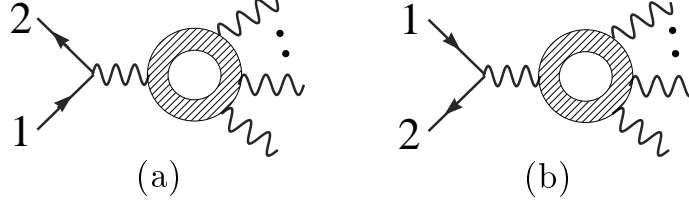


Figure 4. In diagram (a) the external fermion line passes to the ‘left’ of the loop, and in diagram (b) it passes to the ‘right’. A gluon, fermion or scalar may circulate in the closed loop (hatched region).

The ‘left’ and ‘right’ diagrams have to be separated from each other when the external fermions are in the fundamental representation because their color factors are different. (For gluinos in the adjoint representation the color factors are identical, and in fact the ‘left’ and ‘right’ diagrams must be added together to get supersymmetric partial amplitudes.) We shall show below that this division into ‘left’ diagrams and ‘right’ diagrams is gauge invariant; those ‘left’ and ‘right’ diagrams with a closed fermion or scalar loop form further gauge-invariant sets. We thus take the primitive amplitudes for $\bar{q}qg \dots g$ to be

$$\begin{aligned} A_n^{L,[J]}(1_{\bar{q}}, 3, 4, \dots, 2_q, \dots, n), \\ A_n^{R,[J]}(1_{\bar{q}}, 3, 4, \dots, 2_q, \dots, n), \end{aligned} \quad J = 1, \frac{1}{2}, 0, \quad (3.1)$$

corresponding to the sum of all diagrams with the indicated cyclic ordering of external legs, where the fermion line from $1_{\bar{q}}$ to 2_q passes to the left (L) or to the right (R) of the loop, and where $J = \frac{1}{2}$ ($J = 0$) represents the subset of diagrams with a closed fermion loop (closed scalar loop). The normalization is such that two helicity states (Weyl fermions or complex scalars) circulate in the loop. Diagrams without closed fermion or scalar loops are assigned to $J = 1$; they may or may not contain a closed gluon loop, as the two types of diagrams mix under gauge transformations. We shall often suppress the superscript “[1]”; this creates no ambiguity. In the next section and in appendix I we show that all of the quark-gluon partial amplitudes $A_{n;j}$ can be obtained from these primitive amplitudes.

Not all of the primitive amplitudes (3.1) are independent. By flipping over the set of diagrams where $1_{\bar{q}}$ turns right, one obtains (up to a sign) the set of diagrams where $1_{\bar{q}}$ turns left, with the cyclic ordering also reversed,

$$A_n^{R,[J]}(1_{\bar{q}}, 3, 4, \dots, 2_q, \dots, n-1, n) = (-1)^n A_n^{L,[J]}(1_{\bar{q}}, n, n-1, \dots, 2_q, \dots, 4, 3). \quad (3.2)$$

Also, the super-Yang-Mills partial amplitudes for two gluinos and $n-2$ gluons $A_n^{\text{SUSY}} \equiv A_{n;1}^{\text{SUSY}}$ are given by the sum (with all cyclic orderings identical)

$$A_n^{\text{SUSY}} \equiv A_{n;1}^{\text{SUSY}} = A_n^L + A_n^R + A_n^{L,[1/2]} + A_n^{R,[1/2]}, \quad (3.3)$$

because the ‘left’ and ‘right’ diagrams have the same group-theory weight for an adjoint-representation fermion. In this equation supersymmetric cancellations occur between the ‘left’ and ‘right’ primitive amplitudes; in general, A_n^{SUSY} is a much simpler quantity than either A_n^L or A_n^R . Equation (3.3) allows one to obtain one of the four terms on the right for free (say A_n^R), given $A_{n;1}^{\text{SUSY}}$.

Finally, the following fermion-loop contributions vanish,

$$\begin{aligned} A_n^{R,[1/2]}(1_{\bar{q}}, 2_q, 3, 4, \dots, n) &= A_n^{L,[1/2]}(1_{\bar{q}}, n, \dots, 4, 3, 2_q) = 0, \\ A_n^{R,[1/2]}(1_{\bar{q}}, 3, 2_q, 4, \dots, n) &= A_n^{L,[1/2]}(1_{\bar{q}}, n, \dots, 4, 2_q, 3) = 0, \end{aligned} \quad (3.4)$$

and similarly for the scalar-loop contributions. The restriction to ‘left’ or ‘right’ diagrams combines with the ordering of the external legs to leave only tadpole and massless external bubble diagrams behind; but these are zero in dimensional regularization.

We conclude this section by proving that the primitive amplitudes (3.1) are gauge-invariant.[†] Suppose that the two external fermions (f_1 and f_2) are cyclicly separated from each other by n_a gluons going one way around the loop, and by n_b gluons going the other way around ($n_a + n_b = n - 2$), and set aside for a moment the closed fermion- or scalar-loop contributions. The sum $A_n^L + A_n^R$ is obviously gauge invariant because it equals the gauge-invariant, color-ordered partial amplitude $A_{n;1}^{\text{SUSY}} - A_{n;1}^{[1/2]}$, i.e. the coefficient of the leading color structure

$$N_c \text{Tr}(T^{f_1} T^{a_1} \dots T^{a_{n_a}} T^{f_2} T^{b_1} \dots T^{b_{n_b}}) \quad (3.5)$$

in a theory where the fermion line is in the adjoint representation, and omitting the fermion loop contribution. (Different color structures are orthogonal in the large N_c limit, so their coefficients, when independent of N_c , must be individually gauge invariant [6].)

To prove that A_n^L and A_n^R are each invariant independently, we consider a different gauge theory, with gauge group $SU(N_a) \times SU(N_b)$, where the n_a gluons belong to $SU(N_a)$, the n_b gluons belong to $SU(N_b)$, fermion 2 belongs to the representation (N_a, N_b) , and anti-fermion 1 belongs to the representation $(\overline{N}_a, \overline{N}_b)$. The coefficients of the color structures

$$N_a (T^{a_{n_a}} \dots T^{a_1})_{i_2}^{\bar{i}_1} (T^{b_1} \dots T^{b_{n_b}})_{i'_2}^{\bar{i}'_1} \quad (3.6)$$

and

$$N_b (T^{a_{n_a}} \dots T^{a_1})_{i_2}^{\bar{i}_1} (T^{b_1} \dots T^{b_{n_b}})_{i'_2}^{\bar{i}'_1} \quad (3.7)$$

must be separately gauge invariant, since N_a and N_b can be chosen independently. But the coefficient of the first of these color structures is given precisely by the set of ‘right’ diagrams, so it is just

[†] One can motivate the argument using string theory, because each primitive amplitude arises from a distinct string sector. Assuming that the complete set of string states is inserted between gluon n and fermion 1, then A_n^L arises from the Neveu-Schwarz sector and A_n^R from the Ramond sector.

A_n^R . This result holds because when the internal gluon line runs past the n_a gluons from $SU(N_a)$, and the fermion line runs past the n_b gluons from $SU(N_b)$, then the internal gluon must belong to $SU(N_a)$, and an extra factor of N_a is generated in each such graph. Similarly, in the case that the internal gluon line runs past the n_b gluons from $SU(N_b)$, an extra factor of N_b is generated in each graph, and so the second coefficient is A_n^L . Thus A_n^L and A_n^R are separately gauge invariant. This argument works even if there are no external gluons between fermions f_1 and f_2 , that is, $n_a = 0$. A similar argument can be carried out for the fermion (scalar) loop contributions $A_n^{L,[1/2]}$ (respectively $A_n^{L,[0]}$) simply by adding some extra fermion (scalar) flavors transforming under $SU(N_a)$ but not under $SU(N_b)$, in order to separate the L and R diagrams.

4. From Primitive Amplitudes to Partial Amplitudes

In this section, we present the relation of the $\bar{q}qg \dots g$ partial amplitudes $A_{n;j}$ to the primitive amplitudes $A_n^{L,[J]}$, $A_n^{R,[J]}$. The easiest case is the leading partial amplitude $A_{n;1}(1_{\bar{q}}, 2_q, 3, \dots, n)$, the coefficient of the leading color coefficient $N_c (T^{a_3} \dots T^{a_n})_{i_2}^{\bar{i}_1}$. Inspecting color flows in the double-line formalism, it is easy to see that this amplitude receives contributions only from diagrams where the two fermions are adjacent. The $A_n^L(1_{\bar{q}}, 2_q, 3, \dots, n)$ diagrams contribute with weight 1, after noting that the factor of N_c in the diagrams supplies the explicit N_c in the definition (2.5) of $\text{Gr}_{n;1}^{(\bar{q}q)}$. The $A_n^R(1_{\bar{q}}, 2_q, 3, \dots, n)$ diagrams would not contribute at all to $A_{n;1}$, because of their wrong color flow, were it not for the $-1/N_c$ term in the $SU(N_c)$ Fierz identity (2.2). Because of this term, the $A_n^R(1_{\bar{q}}, 2_q, 3, \dots, n)$ diagrams contribute with weight $-1/N_c^2$. The fermion-loop piece $A_n^{L,[1/2]}(1_{\bar{q}}, 2_q, 3, \dots, n)$ contributes with weight n_f/N_c , and similarly for scalar loops. As discussed near eqn. (3.4) the primitive amplitude $A_n^{R,[1/2]}(1_{\bar{q}}, 2_q, 3, \dots, n)$ vanishes because all its diagrams are tadpoles. Putting the contributions together, $A_{n;1}$ is given by

$$\begin{aligned} A_{n;1}(1_{\bar{q}}, 2_q, 3, \dots, n) &= A_n^L(1_{\bar{q}}, 2_q, 3, \dots, n) - \frac{1}{N_c^2} A_n^R(1_{\bar{q}}, 2_q, 3, \dots, n) \\ &+ \frac{n_f}{N_c} A_n^{L,[1/2]}(1_{\bar{q}}, 2_q, 3, \dots, n) + \frac{n_s}{N_c} A_n^{L,[0]}(1_{\bar{q}}, 2_q, 3, \dots, n). \end{aligned} \quad (4.1)$$

One can make use of supersymmetry (3.3) to rewrite (4.1) as

$$\begin{aligned} A_{n;1}(1_{\bar{q}}, 2_q, 3, \dots, n) &= A_n^{\text{SUSY}}(1_{\bar{q}}, 2_q, 3, \dots, n) - \left(1 + \frac{1}{N_c^2}\right) A_n^R(1_{\bar{q}}, 2_q, 3, \dots, n) \\ &+ \left(1 - \frac{n_f}{N_c}\right) A_n^f(1_{\bar{q}}, 2_q, 3, \dots, n) + \left(1 + \frac{n_s}{N_c} - \frac{n_f}{N_c}\right) A_n^s(1_{\bar{q}}, 2_q, 3, \dots, n), \\ &= \left(1 + \frac{1}{N_c^2}\right) A_n^L(1_{\bar{q}}, 2_q, 3, \dots, n) - \frac{1}{N_c^2} A_n^{\text{SUSY}}(1_{\bar{g}}, 2_{\bar{g}}, 3, \dots, n) \\ &- \left(\frac{n_f}{N_c} + \frac{1}{N_c^2}\right) A_n^f(1_{\bar{q}}, 2_q, 3, \dots, n) + \left(\frac{n_s - n_f}{N_c} - \frac{1}{N_c^2}\right) A_n^s(1_{\bar{q}}, 2_q, 3, \dots, n), \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} A_n^f &= -A_n^{L,[0]} - A_n^{L,[1/2]}, \\ A_n^s &= A_n^{L,[0]}. \end{aligned} \quad (4.3)$$

By the expression $A_n^{\text{SUSY}}(1_{\bar{q}}, 2_q, 3, \dots, n)$ we simply mean the supersymmetric $\tilde{g}\tilde{g}g \dots g$ amplitude $A_{n;1}^{\text{SUSY}}(1_{\tilde{g}}, 2_{\tilde{g}}, 3, \dots, n)$ with gluino labels replaced by antiquark/quark labels. The combination A_n^f is simpler than $A_n^{L,[1/2]}$ or $A_n^{L,[0]}$, because it can be viewed as a chiral matter supermultiplet contribution to a supersymmetric amplitude.

In appendix I we prove that the subleading-color partial amplitudes $A_{n;j>1}$ may be expressed as a permutation sum over primitive amplitudes,

$$\begin{aligned} A_{n;j}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, j+3, \dots, n) &= (-1)^{j-1} \sum_{\sigma \in \text{COP}\{\alpha\}\{\beta\}} \left[A_n^{L,[1]}(\sigma(1_{\bar{q}}, 2_q, 3, \dots, n)) \right. \\ &\quad \left. - \frac{n_f}{N_c} A_n^{R,[1/2]}(\sigma(1_{\bar{q}}, 2_q, 3, \dots, n)) - \frac{n_s}{N_c} A_n^{R,[0]}(\sigma(1_{\bar{q}}, 2_q, 3, \dots, n)) \right], \end{aligned} \quad (4.4)$$

where $\alpha_i \in \{\alpha\} \equiv \{j+1, j, \dots, 4, 3\}$, $\beta_i \in \{\beta\} \equiv \{1, 2, j+2, j+3, \dots, n-1, n\}$, and $\text{COP}\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1, 2, \dots, n\}$ with leg 1 held fixed that preserve the cyclic ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta\}$, while allowing for all possible relative orderings of the α_i with respect to the β_i . For example if $\{\alpha\} = \{4, 3\}$ and $\{\beta\} = \{1, 2, 5\}$, then $\text{COP}\{\alpha\}\{\beta\}$ contains the twelve elements

$$\begin{aligned} (1, 2, 5, 4, 3), \quad (1, 2, 4, 5, 3), \quad (1, 4, 2, 5, 3), \quad (1, 2, 4, 3, 5), \quad (1, 4, 3, 2, 5), \quad (1, 4, 2, 3, 5), \\ (1, 2, 5, 3, 4), \quad (1, 2, 3, 5, 4), \quad (1, 3, 2, 5, 4), \quad (1, 2, 3, 4, 5), \quad (1, 3, 4, 2, 5), \quad (1, 3, 2, 4, 5) \end{aligned} \quad (4.5)$$

(cyclic ordering for a two-element set is meaningless). Note that the ordering of the first set of indices is reversed with respect to the second. Formula (4.4) is analogous to the one proven for adjoint representation external states in ref. [20]. (In ref. [20] leg n was held fixed; the choice of fixed leg is completely arbitrary, and here we find it convenient to hold fixed a fermion leg, labeled by 1.)

The terms independent of n_f and n_s in formula (4.4) (or equivalently, formula (I.2)) may be heuristically understood in terms of ‘parent’ diagrams, which have no trees attached to the loop, as depicted in fig. 5. Performing a color decomposition on ordinary Feynman diagrams, using eqns. (2.1) and (2.2), it is easy to see that the set of all parent diagrams feed into both $A_{n;j}$ and $A_n^{L,[1]}$ in the correct way so that eqn. (I.2) is satisfied for this class of diagrams. Roughly speaking, gauge invariance requires all other diagrams to tag along properly with the parent diagrams. The right-hand sides of (4.4) and (I.2) do, however, contain diagrams not appearing on the left-hand side; in appendix I we prove that all such unwanted diagrams cancel in the permutation sum, and also treat the n_f - and n_s -dependent terms.

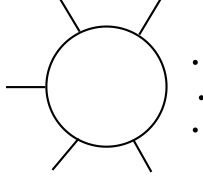


Figure 5. Parent diagrams have no trees attached to the loop. The diagram lines represent any particles in the theory.

Equation (4.4) contains in the permutation sum primitive amplitudes $A_n^L \equiv A_n^{L,[1]}$ with any number of gluons sandwiched between the fermions $1_{\bar{q}}$ and 2_q . If we know the supersymmetric n -gluon partial amplitudes, then we can use the reflection symmetry (3.2) plus supersymmetry (3.3) to eliminate those A_n^L with more than $(n-2)/2$ gluons sandwiched between $1_{\bar{q}}$ and 2_q . Below we carry out this procedure for the four- and five-point cases.

As an explicit example, we present the subleading-color four-point $\bar{q}qgg$ amplitude relations, which are particularly simple,

$$\begin{aligned} A_{4;3}(1_{\bar{q}}, 2_q; 3, 4) &= A_4^L(1_{\bar{q}}, 2_q, 3, 4) + A_4^L(1_{\bar{q}}, 2_q, 4, 3) + A_4^L(1_{\bar{q}}, 3, 2_q, 4) + A_4^L(1_{\bar{q}}, 4, 2_q, 3) \\ &\quad + A_4^L(1_{\bar{q}}, 3, 4, 2_q) + A_4^L(1_{\bar{q}}, 4, 3, 2_q) \\ &= A_{4;1}^{\text{SUSY}}(1_{\bar{q}}, 2_q, 3, 4) + A_{4;1}^{\text{SUSY}}(1_{\bar{q}}, 2_q, 4, 3) + A_{4;1}^{\text{SUSY}}(1_{\bar{q}}, 3, 2_q, 4), \end{aligned} \quad (4.6)$$

where we used eqns. (3.2) and (3.3). In $A_{4;3}$ the fermion and scalar loop contributions (I.5) vanish. The only orderings that do not vanish by equation (3.4) are $A_4^{R,[1/2]}(1_{\bar{q}}, 3, 4, 2_q) + A_4^{R,[1/2]}(1_{\bar{q}}, 4, 3, 2_q)$, and the same combination with $[1/2]$ replaced by $[0]$; these combinations cancel due to Furry's theorem. In fact the expression for $A_{4;3}$ reduces to a supersymmetric quantity. One can verify this relation using the explicit amplitudes given by KST [16].

The five-point relations are of course a bit more complicated,

$$\begin{aligned} A_{5;3}(1_{\bar{q}}, 2_q; 4, 5; 3) &= \sum_{\sigma \in S_3} A_5^L(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^L(1_{\bar{q}}, 4, 5, 2_q, 3) + A_5^L(1_{\bar{q}}, 5, 4, 2_q, 3) \\ &\quad + A_5^L(1_{\bar{q}}, 5, 2_q, 3, 4) + A_5^L(1_{\bar{q}}, 4, 2_q, 5, 3) + A_5^L(1_{\bar{q}}, 4, 2_q, 3, 5) + A_5^L(1_{\bar{q}}, 5, 2_q, 4, 3), \end{aligned} \quad (4.7)$$

$$\begin{aligned} A_{5;4}(1_{\bar{q}}, 2_q; 3, 4, 5) &= \sum_{\sigma \in Z_3} \left[-A_5^L(1_{\bar{q}}, 2_q, \sigma(5), \sigma(4), \sigma(3)) - A_5^L(1_{\bar{q}}, \sigma(5), 2_q, \sigma(4), \sigma(3)) \right. \\ &\quad \left. - A_5^L(1_{\bar{q}}, \sigma(5), \sigma(4), 2_q, \sigma(3)) - A_5^L(1_{\bar{q}}, \sigma(5), \sigma(4), \sigma(3), 2_q) \right. \\ &\quad \left. - \frac{n_s - n_f}{N_c} \left(A_5^s(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^s(1_{\bar{q}}, \sigma(3), 2_q, \sigma(4), \sigma(5)) \right) \right. \\ &\quad \left. + \frac{n_f}{N_c} \left(A_5^f(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^f(1_{\bar{q}}, \sigma(3), 2_q, \sigma(4), \sigma(5)) \right) \right], \end{aligned} \quad (4.8)$$

where A_5^s and A_5^f are defined by equation (4.3). For $A_{5;3}$ the $n_{f,s}$ terms cancel because of equation (3.4) plus Furry's theorem; the contribution with ordering $(1_{\bar{q}}, 4, 5, 2_q, 3)$ cancels the contribution with ordering $(1_{\bar{q}}, 5, 4, 2_q, 3)$. We have also used the reflection identity (3.2) to convert R -type

fermion/scalar loop contributions into L -type ones. We can further use the reflection identity on the $J = 1$ primitive amplitudes, followed by the supersymmetry identity (3.3) combined with Furry's theorem to write these two partial amplitudes as follows,

$$\begin{aligned}
A_{5;3}(1_{\bar{q}}, 2_q; 4, 5; 3) &= \sum_{\sigma \in S_3} [A_5^L(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^L(1_{\bar{q}}, \sigma(3), 2_q, \sigma(4), \sigma(5))] \\
&\quad - A_5^{\text{SUSY}}(1_{\bar{q}}, 3, 2_q, 4, 5) - A_5^{\text{SUSY}}(1_{\bar{q}}, 3, 2_q, 5, 4) , \\
A_{5;4}(1_{\bar{q}}, 2_q; 3, 4, 5) &= - \sum_{\sigma \in S_3} [A_5^L(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^L(1_{\bar{q}}, \sigma(3), 2_q, \sigma(4), \sigma(5))] \\
&\quad + \sum_{\sigma \in Z_3} \left[A_5^{\text{SUSY}}(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^{\text{SUSY}}(1_{\bar{q}}, \sigma(3), 2_q, \sigma(4), \sigma(5)) \right. \\
&\quad \left. - \left[\frac{n_s - n_f}{N_c} - 1 \right] \left(A_5^s(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^s(1_{\bar{q}}, \sigma(3), 2_q, \sigma(4), \sigma(5)) \right) \right. \\
&\quad \left. + \left[\frac{n_f}{N_c} + 1 \right] \left(A_5^f(1_{\bar{q}}, 2_q, \sigma(3), \sigma(4), \sigma(5)) + A_5^f(1_{\bar{q}}, \sigma(3), 2_q, \sigma(4), \sigma(5)) \right) \right] . \tag{4.9}
\end{aligned}$$

In the next section we give explicit formulae for the primitive amplitudes A^{SUSY} , A^L , A^s and A^f with orderings $(1_{\bar{q}}, 2_q, 3, 4, 5)$ and $(1_{\bar{q}}, 2, 3_q, 4, 5)$. The advantage of the form (4.9) is that all terms in the permutation sum may be obtained by a direct relabeling of these primitive amplitudes.

5. Two-Quark Three-Gluon Primitive Amplitudes

In this section, we give explicit formulæ for all the primitive helicity amplitudes for the two-massless-quark three-gluon process $\bar{q}qggg$. Using eqns. (4.2) and (4.9), one can form the partial color-ordered amplitudes. With these in hand, one can construct either the full amplitude using equation (2.4), or the virtual correction to the parton-level cross-section, arising from the color-summed interference of the one-loop amplitude with the tree amplitude, using equation (2.11).

In calculating the primitive amplitudes for $A_{5;1}(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+, 5^+)$ we used a modification of string-based methods for gluons, while for the other helicity configurations we found it convenient to calculate in field theory, drawing on lessons from string theory. In particular, we used field-theory background-field methods [30] (which are an important ingredient in understanding string-based methods in a conventional language [13]). The color-ordered Feynman rules of background-field Feynman gauge lead to a number of calculational improvements besides the obvious reduction in the number of terms in the three-vertex. In particular, supersymmetric decompositions of the amplitude are more evident [3,17,18], and a unitarity-based method [19] that relies on power-counting criteria [20,21] is easier to apply. We used unitarity to calculate certain “cut-constructible” contributions to the primitive amplitudes $A_5^L(1_{\bar{q}}, 2, 3_q, 4, 5)$ which enter the subleading-color partial amplitudes (see

section 1). To perform the required loop integrations we used the integration methods described in refs. [32,33].

We present our results in a convention where all momenta are taken to be outgoing, that is for the process $0 \rightarrow \bar{q}qggg$; helicity conservation along the fermion line thus implies that the two fermion legs must have opposite helicity. Our sign conventions for the primitive amplitudes respect the antisymmetry of the color-ordered rules in fig. 1 as well as the supersymmetry identities in ref. [6]. The overall sign convention for the explicit helicity amplitudes presented here is actually opposite to that usually chosen for fundamental representation quarks; however, the overall sign of the loop helicity amplitude is irrelevant as long as the tree and loop amplitudes use the same convention. An advantage of this choice of signs is that the signs of the tree-level gluino and quark partial amplitudes agree, so that the supersymmetry Ward identities hold without any sign adjustments.

We shall express the primitive amplitudes in terms of the Lorentz inner-products $s_{ij} \equiv (k_i + k_j)^2 = 2k_i \cdot k_j$, and the spinor inner-products $\langle j l \rangle = \langle j^- | l^+ \rangle = \bar{u}_-(k_j)u_+(k_l)$ and $[j l] = \langle j^+ | l^- \rangle = \bar{u}_+(k_j)u_-(k_l)$, where $u_\pm(k)$ is a massless Weyl spinor with momentum k and chirality \pm [8,6].

Discrete symmetries reduce the number of independent primitive amplitudes. Parity reverses all external helicities in a partial amplitude; it is implemented by the “complex conjugation” operation “ \dagger ”, which is the spinor inner-product substitution $\langle j l \rangle \rightarrow [l j]$, $[j l] \rightarrow \langle l j \rangle$, with *no* substitution of $i \rightarrow -i$. For n -gluon amplitudes, parity takes $A \rightarrow A^\dagger$; for two-quark $(n-2)$ -gluon amplitudes, with the above sign conventions, there is an extra minus sign, $A \rightarrow -A^\dagger$. Using parity, we may restrict our attention to two-quark three-gluon amplitudes having either zero or one negative-helicity gluon.

Charge conjugation changes the identity of a quark to an antiquark and vice-versa. Its effect on the primitive amplitudes is to exchange the role of ‘left’ and ‘right’ contributions, $A^L \leftrightarrow A^R$, because the direction of the fermion arrow is reversed. Because the quark and antiquark have opposite helicity, charge conjugation allows us to fix the helicity of the antiquark to be negative.

Finally, equation (4.2) for the leading-color partial amplitude $A_{5;1}$ requires color-orderings where the antiquark and quark are adjacent, while equation (4.9) for the subleading-color partial amplitudes $A_{5;3}$ and $A_{5;4}$ requires also color-orderings with the antiquark and quark separated by one gluon. We thus need to present primitive amplitudes for eight distinct helicity/color configurations: two are infrared-finite, $A(-\bar{q} + q + + +)$ and $A(-\bar{q} + +_q + +)$; and six are infrared-divergent, $A(-\bar{q} + q - + +)$, $A(-\bar{q} + q + - +)$, $A(-\bar{q} + q + + -)$, $A(-\bar{q} - +_q + +)$, $A(-\bar{q} + +_q - +)$ and $A(-\bar{q} + +_q + -)$. Only the latter sextet enter into the next-to-leading order corrections to the two-quark three-gluon process.

We list the required pieces in turn, beginning with the infrared-finite helicity configuration. To construct $A_{5;1}$ we need

$$\begin{aligned}
A_5^{\text{SUSY}}(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^+) &= 0, \\
A_5^L(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^+) &= \frac{i}{32\pi^2} \frac{\langle 12 \rangle [23] \langle 31 \rangle + \langle 14 \rangle [45] \langle 51 \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} + A_5^s(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^+), \\
A_5^s(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^+) &= -\frac{i}{48\pi^2} \left(\frac{\langle 13 \rangle [34] \langle 41 \rangle^2}{\langle 12 \rangle \langle 34 \rangle^2 \langle 45 \rangle \langle 51 \rangle} + \frac{\langle 14 \rangle \langle 24 \rangle [45] \langle 51 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle^2} + \frac{[23] [25]}{[12] \langle 34 \rangle \langle 45 \rangle} \right), \\
A_5^f(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^+) &= 0.
\end{aligned} \tag{5.1}$$

For the construction of the subleading-color partial amplitudes $A_{5;3}$ and $A_{5;4}$ we also need

$$\begin{aligned}
A_5^{\text{SUSY}}(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+, 5^+) &= 0, \\
A_5^L(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+, 5^+) &= \frac{i}{32\pi^2} \left[\frac{\langle 13 \rangle \langle 14 \rangle [45]}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} - \frac{\langle 13 \rangle^2 [23]}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right] \\
&\quad + A_5^s(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+, 5^+), \\
A_5^s(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+, 5^+) &= \frac{i}{48\pi^2} \frac{\langle 15 \rangle \langle 14 \rangle [45]}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2}, \\
A_5^f(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+, 5^+) &= 0.
\end{aligned} \tag{5.2}$$

The tree amplitude vanishes for this helicity configuration.

The remaining helicity amplitudes are infrared-divergent, and also require an ultraviolet subtraction. We will present the formulæ for unsubtracted amplitudes; to carry out the $\overline{\text{MS}}$ subtraction scheme, one should subtract from the leading-color partial amplitudes $A_{5;1}$ the quantity

$$c_\Gamma \left[\frac{3}{2} \frac{1}{\epsilon} \left(\frac{11}{3} - \frac{2}{3} \frac{n_f}{N_c} - \frac{1}{3} \frac{n_s}{N_c} \right) \right] A_5^{\text{tree}}, \tag{5.3}$$

where

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{5.4}$$

and $D = 4 - 2\epsilon$.

We present our results using the dimensional-reduction variant of dimensional regularization, with the external gluons treated in four dimensions. This scheme is equivalent at one-loop to the string-based ‘four-dimensional helicity’ scheme of ref. [12]. To convert these results to the ‘t Hooft-Veltman scheme, one must add to $A_{5;1}$ the quantity

$$\delta_5 = -c_\Gamma \frac{1}{2} \left(1 - \frac{1}{N_c^2} \right) A_5^{\text{tree}}, \tag{5.5}$$

and modify the coupling constant appropriately [16,12]. We obtained the quantity (5.5) by direct calculation in the different schemes, noting that only the singular terms in the integrals contribute

to this quantity. This shift is the same as the one found by KST [16] for four-point $\bar{q}qgg$ amplitudes. The universality of the shift $\delta_n = -c_\Gamma \frac{1}{2}(1 - \frac{1}{N_c^2}) A_n^{\text{tree}}$ for the two-quark $(n-2)$ -gluon amplitudes $A_{n;1}$ can be inferred from the invariance of physical cross-sections under scheme shifts [16]. It can also be inferred from the universality of collinear limits (see eqn. (III.1)). The pure gluon loop splitting amplitudes do not depend on which scheme is used [20]; thus in equation (III.1) a shift δ_{n-1} for $A_{n-1;1}$ of the form (5.5) implies a shift δ_n of exactly the same form for $A_{n;1}$. One may also convert the expressions to conventional dimensional regularization. To do so one must account for the difference between conventional and 't Hooft-Veltman schemes by having $[\epsilon]$ -helicities (gluon polarizations pointing into the -2ϵ dimensions) in observable legs [34]. Since the amplitudes with $[\epsilon]$ -helicities contain an explicit overall ϵ only the universal poles in ϵ enter and the scheme differences may be expected to affect only terms proportional to the tree-level matrix elements. The conversion between the various schemes is discussed in ref. [16].

The cancellation of infrared (soft and collinear) divergences only occurs after combining the virtual corrections presented here with tree-level six-parton contributions to the full next-to-leading order process. Various general formalisms exist for constructing infrared-finite distributions numerically [35,7].

For the infrared-divergent amplitudes, it is convenient to decompose the primitive amplitudes further in a manner analogous to the decomposition of the five-gluon amplitudes [3],

$$A^x = c_\Gamma (V^x A_5^{\text{tree}} + iF^x), \quad x = \text{SUSY}, L, s, f. \quad (5.6)$$

The V factors are purely functions of the momentum invariants $s_{i,i+1} = (k_i + k_{i+1})^2$, and do not contain other spinor products. All the poles in ϵ are contained in the V factors. There is of course some freedom in shifting finite terms between the V and F terms. For the supersymmetric component, the V factor is given by a linear combination of V functions for the all-gluon amplitude [3] (after adjusting for the $\overline{\text{MS}}$ subtraction),

$$V^{\text{SUSY}} = V_{\text{gluon}}^g + 3V_{\text{gluon}}^f, \quad (5.7)$$

while F^{SUSY} is related to the all-gluon F terms by a supersymmetry Ward identity. The function V_{gluon}^g is independent of helicities [3],

$$V_{\text{gluon}}^g = -\frac{1}{\epsilon^2} \sum_{j=1}^5 \left(\frac{\mu^2}{-s_{j,j+1}} \right)^\epsilon + \sum_{j=1}^5 \ln \left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}} \right) \ln \left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}} \right) + \frac{5}{6} \pi^2. \quad (5.8)$$

It will be convenient for us to define a related helicity-independent function with the clockwise set of double poles from the \bar{q} to the q omitted,

$$V_{\bar{q}q}^g = V_{\text{gluon}}^g + \sum_{j=\bar{q}}^{q-1} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{j,j+1}} \right)^\epsilon. \quad (5.9)$$

All the scalar- and fermion-loop primitive amplitudes turn out to be free of poles, so that we may take

$$V^s = V^f = 0 . \quad (5.10)$$

We denote the dilogarithm [36] by Li_2 ,

$$\text{Li}_2(x) = - \int_0^x dy \frac{\ln(1-y)}{y} . \quad (5.11)$$

In order to present the results for the remaining functions, we also define the following functions [3],

$$\begin{aligned} L_0(r) &= \frac{\ln(r)}{1-r} , & L_1(r) &= \frac{L_0(r) + 1}{1-r} , & L_2(r) &= \frac{\ln(r) - \frac{1}{2}(r - 1/r)}{(1-r)^3} , \\ \text{LS}_{-1}(r_1, r_2) &= \text{Li}_2(1-r_1) + \text{Li}_2(1-r_2) + \ln r_1 \ln r_2 - \frac{\pi^2}{6} , \\ \text{LS}_0(r_1, r_2) &= \frac{1}{(1-r_1-r_2)} [\text{LS}_{-1}(r_1, r_2)] , \\ \text{LS}_1(r_1, r_2) &= \frac{1}{(1-r_1-r_2)} [\text{LS}_0(r_1, r_2) + L_0(r_1) + L_0(r_2)] , \\ \text{LS}_2(r_1, r_2) &= \frac{1}{(1-r_1-r_2)} \left[\text{LS}_1(r_1, r_2) + \frac{1}{2} (L_1(r_1) + L_1(r_2)) \right] , \\ \text{LS}_3(r_1, r_2) &= \frac{1}{(1-r_1-r_2)} \left[\text{LS}_2(r_1, r_2) + \frac{1}{3} (L_2(r_1) + L_2(r_2)) - \frac{1}{6r_1} - \frac{1}{6r_2} \right] . \end{aligned} \quad (5.12)$$

The L_i functions are nonsingular as $r \rightarrow 1$, and the LS_i functions are nonsingular as $1-r_1-r_2 \rightarrow 0$.

For $A_5(1_{\bar{q}}, 2_q^+, 3^-, 4^+, 5^+)$ the functions needed in order to construct the amplitude via equations (4.2), (5.6) are:

$$\begin{aligned} A_5^{\text{tree}} &= i \frac{\langle 13 \rangle^3 \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} , \\ V^{\text{SUSY}} &= V_{\text{gluon}}^g - \frac{3}{2\epsilon} \left(\left(\frac{\mu^2}{-s_{34}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{51}} \right)^\epsilon \right) - 6 , \\ V^L &= V_{12}^g - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{34}} \right)^\epsilon + \ln \left(\frac{-s_{51}}{-s_{12}} \right) - 3 , \end{aligned} \quad (5.13)$$

$$\begin{aligned} F^{\text{SUSY}} &= 3 \frac{\langle 32 \rangle}{\langle 31 \rangle} F_{\text{gluon}}^f \\ &= -3 \frac{\langle 13 \rangle \langle 23 \rangle \langle 41 \rangle [24]^2}{\langle 45 \rangle \langle 51 \rangle} \frac{\text{LS}_1 \left(\frac{-s_{23}}{-s_{51}}, \frac{-s_{34}}{-s_{51}} \right)}{s_{51}^2} + 3 \frac{\langle 13 \rangle \langle 23 \rangle \langle 53 \rangle [25]^2}{\langle 34 \rangle \langle 45 \rangle} \frac{\text{LS}_1 \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right)}{s_{34}^2} \\ &\quad - \frac{3}{2} \frac{\langle 13 \rangle^2}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \left(\langle 15 \rangle [52] \langle 23 \rangle + \langle 12 \rangle [24] \langle 43 \rangle \right) \frac{L_0 \left(\frac{-s_{34}}{-s_{51}} \right)}{s_{51}} , \end{aligned}$$

$$\begin{aligned}
F^L = F^s & - \frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [24]^3}{\langle 45 \rangle \langle 51 \rangle} \frac{\text{Ls}_2 \left(\frac{-s_{23}}{-s_{51}}, \frac{-s_{34}}{-s_{51}} \right)}{s_{51}^3} - \frac{\langle 12 \rangle \langle 23 \rangle \langle 35 \rangle^2 [25]^3}{\langle 34 \rangle \langle 45 \rangle} \frac{\text{Ls}_2 \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right)}{s_{34}^3} \\
& - 2 \frac{\langle 13 \rangle \langle 23 \rangle \langle 41 \rangle [24]^2}{\langle 45 \rangle \langle 51 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{23}}{-s_{51}}, \frac{-s_{34}}{-s_{51}} \right)}{s_{51}^2} - 2 \frac{\langle 13 \rangle \langle 23 \rangle \langle 35 \rangle [25]^2}{\langle 34 \rangle \langle 45 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right)}{s_{34}^2} \\
& - \frac{\langle 13 \rangle^2 [24]}{\langle 45 \rangle \langle 51 \rangle} \frac{\text{Ls}_0 \left(\frac{-s_{23}}{-s_{51}}, \frac{-s_{34}}{-s_{51}} \right)}{s_{51}} - \frac{\langle 13 \rangle^2 \langle 35 \rangle [25]}{\langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \frac{\text{Ls}_0 \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right)}{s_{34}} \\
& - \left(\frac{\langle 13 \rangle \langle 23 \rangle^2 [25]^2 \langle 15 \rangle}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle} + \frac{1}{2} \frac{\langle 13 \rangle^2 [12] \langle 23 \rangle [25]}{\langle 34 \rangle \langle 45 \rangle} \right) \frac{\text{L}_1 \left(\frac{-s_{34}}{-s_{51}} \right)}{s_{51}^2} \\
& + \frac{1}{2} \frac{\langle 13 \rangle \langle 14 \rangle \langle 23 \rangle [24]^2}{\langle 45 \rangle \langle 51 \rangle} \frac{\text{L}_1 \left(\frac{-s_{23}}{-s_{51}} \right)}{s_{51}^2} - \frac{1}{2} \frac{\langle 13 \rangle \langle 15 \rangle \langle 34 \rangle [45]^2}{\langle 12 \rangle \langle 45 \rangle} \frac{\text{L}_1 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} \\
& + \frac{1}{2} \frac{\langle 13 \rangle^2 [24]}{\langle 45 \rangle \langle 51 \rangle} \frac{\text{L}_0 \left(\frac{-s_{34}}{-s_{51}} \right)}{s_{51}} - \left[2 \frac{\langle 13 \rangle^2 [45]}{\langle 12 \rangle \langle 45 \rangle} + \frac{\langle 13 \rangle^2 \langle 35 \rangle [25]}{\langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right] \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}} \\
& + \frac{1}{2} \frac{\langle 14 \rangle [24]^2 [45]}{\langle 45 \rangle [23] [34] s_{51}} - \frac{\langle 13 \rangle \langle 23 \rangle [25] [45]}{\langle 12 \rangle \langle 34 \rangle [34] \langle 45 \rangle [15]} - \frac{1}{2} \frac{\langle 13 \rangle^2 [12] \langle 23 \rangle [25]}{\langle 34 \rangle^2 [34] \langle 45 \rangle \langle 15 \rangle [15]} , \\
F^s & = \frac{1}{3} \left[\frac{\langle 15 \rangle [25] \langle 34 \rangle \langle 35 \rangle [45]^2}{\langle 45 \rangle} \frac{\text{L}_2 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}^3} - \frac{\langle 13 \rangle \langle 15 \rangle \langle 34 \rangle [45]^2}{\langle 12 \rangle \langle 45 \rangle} \frac{\text{L}_1 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} \right. \\
& \left. - \frac{\langle 13 \rangle [24] [45]}{\langle 12 \rangle [12] [34] \langle 45 \rangle} + \frac{[24]^2 [25]}{[12] [23] [34] \langle 45 \rangle} \right] , \\
F^f & = - \frac{\langle 13 \rangle^2 [45]}{\langle 12 \rangle \langle 45 \rangle} \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}} .
\end{aligned} \tag{5.14}$$

For $A_5(1_q^-, 2_q^+, 3^+, 4^-, 5^+)$ the various functions are

$$\begin{aligned}
A_5^{\text{tree}} & = i \frac{\langle 14 \rangle^3 \langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} , \\
V^{\text{SUSY}} & = V_{\text{gluon}}^g - \frac{3}{2\epsilon} \left(\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{34}} \right)^\epsilon \right) - 6 , \\
V^L & = V_{12}^g - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{34}} \right)^\epsilon - 2 ,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
F^{\text{SUSY}} & = 3 \frac{\langle 42 \rangle}{\langle 41 \rangle} F_{\text{gluon}}^f \\
& = -3 \frac{\langle 13 \rangle \langle 14 \rangle \langle 24 \rangle [35]^2}{\langle 12 \rangle \langle 23 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}^2} - 3 \frac{\langle 14 \rangle \langle 24 \rangle^2 [25]^2}{\langle 23 \rangle \langle 34 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{51}}{-s_{34}}, \frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} \\
& \quad - \frac{3}{2} \frac{\langle 14 \rangle^2 \langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \left(\langle 12 \rangle [25] \langle 54 \rangle + \langle 15 \rangle [53] \langle 34 \rangle \right) \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}} ,
\end{aligned}$$

$$\begin{aligned}
F^L = F^s & - \frac{\langle 13 \rangle \langle 34 \rangle \langle 45 \rangle [35]^3}{\langle 23 \rangle} \frac{\text{Ls}_2 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}^3} - \frac{\langle 12 \rangle \langle 24 \rangle^2 \langle 45 \rangle [25]^3}{\langle 23 \rangle \langle 34 \rangle} \frac{\text{Ls}_2 \left(\frac{-s_{51}}{-s_{34}}, \frac{-s_{12}}{-s_{34}} \right)}{s_{34}^3} \\
& + \frac{\langle 14 \rangle [35]^2 (\langle 14 \rangle \langle 23 \rangle \langle 15 \rangle + 2 \langle 13 \rangle \langle 15 \rangle \langle 24 \rangle + \langle 13 \rangle \langle 14 \rangle \langle 25 \rangle)}{\langle 12 \rangle \langle 23 \rangle \langle 51 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}^2} \\
& + \frac{\langle 14 \rangle \langle 24 \rangle [25]^2 (2 \langle 15 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 25 \rangle)}{\langle 23 \rangle \langle 34 \rangle \langle 51 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{51}}{-s_{34}}, \frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} + \frac{\langle 14 \rangle^3 \langle 13 \rangle [13]^2}{\langle 23 \rangle \langle 45 \rangle \langle 51 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{12}}{-s_{45}}, \frac{-s_{23}}{-s_{45}} \right)}{s_{45}^2} \\
& - \frac{3}{2} \frac{\langle 14 \rangle^2 \langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \left(\langle 12 \rangle [25] \langle 54 \rangle + \langle 15 \rangle [53] \langle 34 \rangle \right) \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}} - \frac{\langle 14 \rangle^2 [35]}{\langle 23 \rangle \langle 15 \rangle} \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}} \\
& + \frac{\langle 14 \rangle^2 \langle 24 \rangle [23]}{\langle 45 \rangle \langle 15 \rangle \langle 23 \rangle} \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}} - \frac{\langle 14 \rangle^3 [13]}{\langle 23 \rangle \langle 45 \rangle \langle 15 \rangle} \frac{\text{L}_0 \left(\frac{-s_{23}}{-s_{45}} \right)}{s_{45}} - \frac{\langle 14 \rangle^2 \langle 24 \rangle [25]}{\langle 23 \rangle \langle 34 \rangle \langle 15 \rangle} \frac{\text{L}_0 \left(\frac{-s_{34}}{-s_{51}} \right)}{s_{51}} \\
& + \frac{1}{2} \left[\frac{\langle 14 \rangle \langle 34 \rangle [35]^2}{\langle 23 \rangle} \frac{\text{L}_1 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} + \frac{\langle 12 \rangle \langle 14 \rangle \langle 24 \rangle \langle 45 \rangle [25]^2}{\langle 23 \rangle \langle 34 \rangle \langle 15 \rangle} \frac{\text{L}_1 \left(\frac{-s_{34}}{-s_{12}} \right)}{s_{12}^2} \right. \\
& + \frac{\langle 14 \rangle \langle 24 \rangle^2 [25]^2}{\langle 23 \rangle \langle 34 \rangle} \frac{\text{L}_1 \left(\frac{-s_{34}}{-s_{51}} \right)}{s_{51}^2} + \frac{\langle 24 \rangle [25]^2 [35]}{[12] \langle 23 \rangle s_{34} [45]} + \frac{\langle 13 \rangle \langle 14 \rangle [15] [35]^2}{s_{12} \langle 23 \rangle s_{34} [45]} + \frac{\langle 14 \rangle \langle 24 \rangle^2 [25]^2}{\langle 23 \rangle \langle 34 \rangle s_{34} s_{51}} \\
& \left. - \frac{\langle 14 \rangle^2 \langle 24 \rangle [23]}{s_{12} \langle 23 \rangle \langle 45 \rangle \langle 15 \rangle} \right], \\
F^s & = 2 \frac{\langle 13 \rangle \langle 34 \rangle \langle 45 \rangle \langle 15 \rangle [35]^4}{\langle 12 \rangle} \frac{\text{Ls}_3 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}^4} \\
& + \frac{2}{3} \frac{\langle 14 \rangle \langle 15 \rangle \langle 34 \rangle [35]^3}{\langle 12 \rangle} \frac{\text{L}_2 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}^3} + \frac{2}{3} \frac{\langle 13 \rangle \langle 14 \rangle \langle 45 \rangle [35]^3}{\langle 12 \rangle} \frac{\text{L}_2 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}^3} - \frac{1}{3} \frac{\langle 13 \rangle \langle 15 \rangle [35]^4}{\langle 12 \rangle [34] [45] s_{12}^2}, \\
F^f & = - \frac{\langle 14 \rangle^2 [35]^2}{\langle 12 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}^2}.
\end{aligned} \tag{5.16}$$

For $A_5(1_q^-, 2_q^+, 3^+, 4^+, 5^-)$ the various functions are

$$\begin{aligned}
A_5^{\text{tree}} & = i \frac{\langle 15 \rangle^3 \langle 25 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}, \\
V^{\text{SUSY}} & = V_{\text{gluon}}^g - \frac{3}{2\epsilon} \left(\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{45}} \right)^\epsilon \right) - 6, \\
V^L & = V_{12}^g - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{45}} \right)^\epsilon - \frac{5}{2}, \\
F^{\text{SUSY}} & = 3 \frac{\langle 52 \rangle}{\langle 51 \rangle} F_{\text{gluon}}^f \\
& = \frac{3}{2} \frac{\langle 15 \rangle \langle 25 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \left(\langle 54 \rangle [43] \langle 31 \rangle + \langle 53 \rangle [32] \langle 21 \rangle \right) \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}},
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
F^L &= F^s - \frac{\langle 15 \rangle^2 \langle 14 \rangle [14]^2}{\langle 23 \rangle \langle 34 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{45}}{-s_{23}}, \frac{-s_{51}}{-s_{23}} \right)}{s_{23}^2} - \frac{\langle 15 \rangle^2 \langle 35 \rangle \langle 13 \rangle [13]^2}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{12}}{-s_{45}}, \frac{-s_{23}}{-s_{45}} \right)}{s_{45}^2} \\
&\quad - \frac{\langle 15 \rangle^2 [14]}{\langle 23 \rangle \langle 34 \rangle} \frac{\text{L}_0 \left(\frac{-s_{23}}{-s_{51}} \right)}{s_{51}} + \frac{\langle 15 \rangle^2 [12] \langle 25 \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \frac{\text{L}_0 \left(\frac{-s_{23}}{-s_{45}} \right)}{s_{45}} \\
&\quad - \left[\frac{\langle 15 \rangle^2 [34]}{\langle 12 \rangle \langle 34 \rangle} - \frac{1}{2} \frac{\langle 15 \rangle \langle 25 \rangle (3 \langle 54 \rangle [43] \langle 31 \rangle + \langle 53 \rangle [32] \langle 21 \rangle)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \right] \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}} \\
&\quad + \frac{1}{2} \left[- \frac{\langle 13 \rangle \langle 35 \rangle \langle 45 \rangle [34]^2}{\langle 23 \rangle \langle 34 \rangle} \frac{\text{L}_1 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}^2} + \frac{[24] [34]}{\langle 34 \rangle [45] [15]} \right] , \\
F^s &= \frac{1}{3} \left[- \frac{\langle 13 \rangle^2 \langle 45 \rangle^2 [34]^3}{\langle 12 \rangle \langle 34 \rangle} \frac{\text{L}_2 \left(\frac{-s_{45}}{-s_{12}} \right)}{s_{12}^3} + \frac{\langle 13 \rangle \langle 45 \rangle \langle 15 \rangle [34]^2}{\langle 12 \rangle \langle 34 \rangle} \frac{\text{L}_1 \left(\frac{-s_{45}}{-s_{12}} \right)}{s_{12}^2} \right. \\
&\quad \left. + 2 \frac{\langle 15 \rangle^2 [34]}{\langle 12 \rangle^2 [12] \langle 34 \rangle} + \frac{\langle 15 \rangle [13] [24]}{\langle 12 \rangle [12] \langle 34 \rangle [15]} + \frac{\langle 14 \rangle [14] [24] [34]}{\langle 12 \rangle [12] \langle 34 \rangle [45] [15]} \right] , \\
F^f &= - \frac{\langle 15 \rangle^2 [34]}{\langle 12 \rangle \langle 34 \rangle} \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}} .
\end{aligned} \tag{5.18}$$

For use in constructing the subleading-color partial amplitudes, the V and F functions for the components of $A_5(1_{\bar{q}}^-, 2^-, 3_q^+, 4^+, 5^+)$ are

$$\begin{aligned}
A_5^{\text{tree}} &= i \frac{\langle 12 \rangle^3 \langle 32 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} , \\
V^{\text{SUSY}} &= V_{\text{gluon}}^g - \frac{3}{2\epsilon} \left(\left(\frac{\mu^2}{-s_{23}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{51}} \right)^\epsilon \right) - 6 , \\
V^L &= V_{13}^g - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon + \text{Ls}_{-1} \left(\frac{-s_{34}}{-s_{51}}, \frac{-s_{23}}{-s_{51}} \right) + \text{Ls}_{-1} \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right) - 3 ,
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
F^{\text{SUSY}} &= 3 \frac{\langle 32 \rangle}{\langle 12 \rangle} F_{\text{gluon}}^f \\
&= \frac{3}{2} \frac{\langle 12 \rangle (\langle 23 \rangle [34] \langle 41 \rangle + \langle 24 \rangle [45] \langle 51 \rangle)}{\langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \frac{\text{L}_0 \left(\frac{-s_{23}}{-s_{51}} \right)}{s_{51}} , \\
F^L &= F^s + \frac{1}{2} \frac{\langle 15 \rangle [54]^2 \langle 42 \rangle}{\langle 34 \rangle \langle 45 \rangle} \frac{\text{L}_1 \left(\frac{-s_{23}}{-s_{51}} \right)}{s_{51}^2} + 2 \frac{\langle 12 \rangle \langle 24 \rangle [45] \langle 51 \rangle}{\langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \frac{\text{L}_0 \left(\frac{-s_{23}}{-s_{51}} \right)}{s_{51}} \\
&\quad + \frac{1}{2} \frac{[45] [35] [13]}{[23] [12] [15] \langle 45 \rangle} , \\
F^s &= \frac{1}{3} \frac{[34] [35]}{[12] [23] \langle 45 \rangle} , \\
F^f &= 0 .
\end{aligned} \tag{5.20}$$

For $A_5(1_{\bar{q}}^-, 2^+, 3_q^+, 4^-, 5^+)$, the functions are

$$\begin{aligned}
A_5^{\text{tree}} &= i \frac{\langle 14 \rangle^3 \langle 34 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} , \\
V^{\text{SUSY}} &= V_{\text{gluon}}^g - \frac{3}{2\epsilon} \left[\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{34}} \right)^\epsilon \right] - 6 , \\
V^L &= V_{13}^g - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{34}} \right)^\epsilon - \frac{1}{2} \ln \left(\frac{-s_{12}}{-s_{34}} \right) + \text{Ls}_{-1} \left(\frac{-s_{23}}{-s_{45}}, \frac{-s_{12}}{-s_{45}} \right) - 3 ,
\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}
F^{\text{SUSY}} &= 3 \frac{\langle 34 \rangle}{\langle 14 \rangle} F_{\text{gluon}}^f \\
&= -3 \frac{\langle 13 \rangle \langle 14 \rangle \langle 34 \rangle [35]^2}{\langle 12 \rangle \langle 23 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{45}}{-s_{12}}, \frac{-s_{34}}{-s_{12}} \right)}{s_{12}^2} - 3 \frac{\langle 14 \rangle \langle 24 \rangle [25]^2}{\langle 23 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{51}}{-s_{34}}, \frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} \\
&\quad + \frac{3}{2} \frac{\langle 14 \rangle^2 \langle 34 \rangle (\langle 12 \rangle [25] \langle 54 \rangle + \langle 15 \rangle [53] \langle 34 \rangle)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 15 \rangle} \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}} , \\
F^L &= - \frac{\langle 13 \rangle^2 \langle 34 \rangle \langle 45 \rangle [35]^3}{\langle 12 \rangle \langle 23 \rangle} \frac{\text{Ls}_2 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}^3} - \frac{\langle 12 \rangle \langle 45 \rangle \langle 24 \rangle [25]^3}{\langle 23 \rangle} \frac{\text{Ls}_2 \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right)}{s_{34}^3} \\
&\quad - 2 \frac{\langle 14 \rangle \langle 34 \rangle \langle 13 \rangle [35]^2}{\langle 12 \rangle \langle 23 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}^2} - 2 \frac{\langle 14 \rangle \langle 24 \rangle [25]^2}{\langle 23 \rangle} \frac{\text{Ls}_1 \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right)}{s_{34}^2} \\
&\quad - \frac{\langle 14 \rangle^2 \langle 13 \rangle [35]}{\langle 12 \rangle \langle 23 \rangle \langle 51 \rangle} \frac{\text{Ls}_0 \left(\frac{-s_{34}}{-s_{12}}, \frac{-s_{45}}{-s_{12}} \right)}{s_{12}} - \frac{\langle 14 \rangle^2 [25]}{\langle 23 \rangle \langle 51 \rangle} \frac{\text{Ls}_0 \left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}} \right)}{s_{34}} \\
&\quad - \frac{1}{2} \frac{\langle 14 \rangle \langle 24 \rangle [25]^2}{\langle 23 \rangle} \frac{\text{L}_1 \left(\frac{-s_{51}}{-s_{34}} \right)}{s_{34}^2} + \frac{1}{2} \frac{\langle 12 \rangle \langle 14 \rangle \langle 45 \rangle [15] [25]}{\langle 23 \rangle \langle 25 \rangle} \frac{\text{L}_1 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} \\
&\quad + \frac{1}{2} \frac{\langle 13 \rangle \langle 14 \rangle \langle 34 \rangle [35]^2}{\langle 12 \rangle \langle 23 \rangle} \frac{\text{L}_1 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}^2} - \left[2 \frac{\langle 14 \rangle^2 \langle 34 \rangle [35]}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle} - \frac{1}{2} \frac{\langle 12 \rangle \langle 14 \rangle \langle 45 \rangle [25]}{\langle 15 \rangle \langle 23 \rangle \langle 25 \rangle} \right] \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{34}} \right)}{s_{34}} \\
&\quad + \frac{1}{2} \frac{\langle 14 \rangle \langle 24 \rangle [25]}{\langle 23 \rangle \langle 25 \rangle \langle 34 \rangle [34]} + \frac{1}{2} \frac{\langle 13 \rangle^2 [35]^3}{s_{12} \langle 12 \rangle \langle 23 \rangle [34] [45]} , \\
F^s &= 0 , \\
F^f &= 0 .
\end{aligned} \tag{5.22}$$

For $A_5(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+, 5^-)$, the functions are

$$\begin{aligned}
A_5^{\text{tree}} &= i \frac{\langle 15 \rangle^3 \langle 35 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} , \\
V^{\text{SUSY}} &= V_{\text{gluon}}^g - \frac{3}{2\epsilon} \left(\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{45}} \right)^\epsilon \right) - 6 , \\
V^L &= V_{13}^g - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \text{Ls}_{-1} \left(\frac{-s_{23}}{-s_{45}}, \frac{-s_{12}}{-s_{45}} \right) - 3 ,
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
F^{\text{SUSY}} &= 3 \frac{\langle 35 \rangle}{\langle 15 \rangle} F_{\text{gluon}}^{\text{SUSY}} \\
&= \frac{3}{2} \frac{\langle 15 \rangle \langle 35 \rangle (\langle 54 \rangle [43] \langle 31 \rangle + \langle 53 \rangle [32] \langle 21 \rangle)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}} , \\
F^L &= \frac{\langle 15 \rangle^2 [24]}{\langle 12 \rangle \langle 34 \rangle} \frac{\text{L}_{\text{S}_0} \left(\frac{-s_{34}}{-s_{51}}, \frac{-s_{23}}{-s_{51}} \right)}{s_{51}} + \frac{\langle 15 \rangle^2 \langle 13 \rangle [14]}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} \frac{\text{L}_{\text{S}_0} \left(\frac{-s_{51}}{-s_{23}}, \frac{-s_{45}}{-s_{23}} \right)}{s_{23}} \\
&\quad - \frac{1}{2} \frac{\langle 35 \rangle \langle 45 \rangle [34]^2 \langle 13 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} \frac{\text{L}_1 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}^2} - 2 \frac{\langle 15 \rangle \langle 35 \rangle \langle 13 \rangle [34]}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} \frac{\text{L}_0 \left(\frac{-s_{12}}{-s_{45}} \right)}{s_{45}} - \frac{1}{2} \frac{[24]^2}{\langle 34 \rangle [15] [45]} , \\
F^S &= 0 , \\
F^f &= 0 .
\end{aligned} \tag{5.24}$$

The remaining amplitudes are related by discrete symmetries to those presented above; consider, for example,

$$A_5^L(1_{\bar{q}}^{\pm}, 2_q^{\mp}, 3^-, 4^-, 5^-):$$

$$\begin{aligned}
A_5^L(1_{\bar{q}}^+, 2_q^-, 3^-, 4^-, 5^-) &= -[A_5^L(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^+)]^{\dagger} \\
A_5^L(1_{\bar{q}}^-, 2_q^+, 3^-, 4^-, 5^-) &= A_5^R(2_q^+, 3^-, 4^-, 5^-, 1_q^-) = -A_5^L(2_q^+, 1_q^-, 5^-, 4^-, 3^-) \\
&= +[A_5^L(2_q^-, 1_q^+, 5^+, 4^+, 3^+)]^{\dagger} .
\end{aligned} \tag{5.25}$$

The overall signs in these relations drop out in the cross-section, because the signs are the same for loop amplitudes and tree amplitudes.

We have performed a variety of checks on the amplitudes:

- 1) a check of collinear factorization for all primitive amplitudes in all channels, illustrated in appendix III, providing an extremely stringent check of the primitive amplitudes;
- 2) a verification of the supersymmetry identities [15] for the primitive amplitudes $A_5(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+, 5^-)$ and $A_5(1_{\bar{q}}^-, 2_q^+, 3^+, 4^-, 5^+)$ by explicitly calculating all terms in the supersymmetry relation (3.3);
- 3) a verification of some of the cuts in amplitudes that were not calculated via cutting methods;
- 4) a check on eqn. (4.9) giving $A_{5;3}$ and $A_{5;4}$, by comparing the poles in ϵ in these quantities against explicit formulae for such singular terms in ref. [28];
- 5) a check of formula (2.11) for the virtual part of the color-summed cross-section, by showing that the infrared poles in ϵ properly cancel against singular terms in the $2 \rightarrow 4$ matrix elements arising from the integration over soft and collinear phase space [35,7,28];
- 6) a check of the permutation formula for $A_{5;4}$ in eqn. (4.9), by comparing the fermion loop (n_f) contributions to the corresponding contribution of the previously calculated [37] process

$Z \rightarrow 3\gamma$, but with the Z polarization vector replaced with a fermion bilinear (and gluon propagator connecting it to the loop); it is not difficult to see that the contributing diagrams are identical for these two cases. (The axial coupling does not contribute to $Z \rightarrow 3\gamma$ so it does not affect the comparison.) We have explicitly verified that the fermion loop contributions on the right-hand side of eqn. (4.9) agree with the appropriately modified expressions contained in ref. [37] for vanishing fermion masses.

6. Conclusions

In this paper, we presented all one-loop QCD amplitudes for two external quarks and three external gluons. Combining these results with the ones for five gluons [3] and four quarks and one gluon [4], this completes the set of one-loop amplitudes required for calculating next-to-leading order corrections to three-jet production at hadron colliders. The computation made use of a number of techniques, including spinor helicity [8], color decompositions [5,9,10], string-based methods [12,14], supersymmetry methods [15,3,17,18], improved gauge choices [30,31,13], perturbative unitarity [20,21], and collinear limits [38,6,23,39].

We also introduced *primitive amplitudes* as gauge-invariant building blocks from which amplitudes containing fundamental representation external legs may be constructed. The usefulness of primitive amplitudes follows from their relatively simple analytic structure. In a previous paper we obtained a formula [20] valid for adjoint representation states which allows one to obtain all subleading-color partial amplitudes from the leading-color partial amplitudes. Using primitive amplitudes we generalized this formula to the case of two external fundamental representation quarks and $(n-2)$ -gluons. Further generalizations to larger numbers of external quarks are straightforward.

In calculating the amplitudes we made extensive use of supersymmetry identities [15], both as a check and to reduce the number of independent amplitudes to be calculated. We verified that as the momenta of two adjacent external legs become collinear, the amplitudes presented here properly reduce to sums of lower-point amplitudes multiplied by universal splitting amplitudes. This provides a stringent consistency check on the amplitudes.

The one-loop two-quark three-gluon amplitudes presented in this paper constitute, along with the five-gluon [3] and four-quark one-gluon [4] ones, one of the major ingredients required for the construction of a next-to-leading order program for the prediction of three-jet physics at hadron colliders. The infrared singularities in these amplitudes must be cancelled by adding the singular contributions from real emission, for example using the formalisms of refs. [7,35]. The program also requires the full form of the real emission contributions in non-singular regions, given here by the known six-point tree amplitudes [5,6]. Such a program would allow the study of three-jet

distributions to next-to-leading order; as with the two-jet case studied extensively by various collider detector collaborations, large-statistics data are available. In the three-jet case, one may study a richer variety of distributions. The comparison of three-jet rates to two-jet rates, in conjunction with such an NLO program, offers the first possibility of measuring the strong coupling constant α_s in a purely hadronic process deep in the perturbative regime. Beyond three-jet studies, such a program also incorporates the elements required to study jet structure beyond the leading non-vanishing order available in two-jet NLO programs.

Acknowledgements.

We thank D.C. Dunbar for useful discussions, and A. Signer for pointing out an error in an earlier version of equation (IV.8); one of us (L.D.) also thanks Z. Kunszt for useful discussions. We are grateful for the support of NATO Collaborative Research Grant CRG-921322 (L.D. and D.A.K.).

Appendix I. Subleading-Color Partial Amplitudes from Primitive Amplitudes

In this appendix we prove that the subleading partial amplitudes $A_{n;j>1}$ are given by sums over permutations of primitive amplitudes. We will use a similar result for the case of n external adjoint particles [20] as an intermediate step in deriving the result for the case that the two external fermions are in the fundamental representation.

First we discuss the distinction between the gauge groups $SU(N_c)$ and $U(N_c) = SU(N_c) \times U(1)$. If all particles in an amplitude transform as the adjoint representation of $SU(N_c)$, and all vertices are given by (combinations of) the structure constants f^{abc} , then there is essentially no distinction, because $f^{abc} = 0$ whenever a corresponds to the $U(1)$ generator, $T^{a_{U(1)}} = \mathbf{1}/\sqrt{N_c}$. In other words, the $U(1)$ ‘photon’ is automatically projected out by vertices such as nonabelian vector-boson self-interactions. The $-1/N_c$ term in the $SU(N_c)$ Fierz identity (2.2) removes the ‘photon’ explicitly by projecting onto traceless hermitian matrices. It can be ignored if only adjoint-representation particles are present.

More generally, the $-1/N_c$ Fierz term only affects those diagrams for $\bar{q}qg \dots g$ where a gauge boson propagator is attached at both ends to a line in the fundamental representation — a fermion or scalar line. In those diagrams where both of these ends are in the loop, that is with exactly one gauge boson propagator in the loop itself (the diagrams contributing to $A_n^R(1_{\bar{q}}, 2_q, 3, \dots, n)$), the $-1/N_c$ term leads to $A_{n;1}$ contributions only, as discussed above. The only other diagrams affected are fermion or scalar loop contributions, where a gauge boson attaches the pinched-off external fermion line to the loop. In summary, the n_f - and n_s -independent parts of the subleading $\bar{q}qg \dots g$ partial amplitudes $A_{n;j>1}$, and all of the $\tilde{g}\tilde{g}g \dots g$ partial amplitudes, can be analyzed as if

the gauge group were $U(N_c)$, neglecting the $-1/N_c$ Fierz term when working out the color flow in the double-line formalism. This result in turn implies that such $\bar{q}qg \dots g$ double-line diagrams can be obtained from a subset of $\tilde{g}\tilde{g}g \dots g$ diagrams by ‘color-stripping’ — removing a color line that flows directly from one gluino to the other, thereby converting the adjoint representation gluino into a fundamental representation quark. Consequently, conversion of a subleading-color formula for $\tilde{g}\tilde{g}g \dots g$ to the $\bar{q}qg \dots g$ case is quite straightforward.

The formula we want to re-derive, and then modify, expresses the subleading-color partial amplitudes $A_{n;j>2}^{\text{adj}}$ for $\tilde{g}\tilde{g}g \dots g$ (or $ggg \dots g$) as a sum over permutations of leading-color partial amplitudes $A_{n;1}$,

$$A_{n;j}^{\text{adj}}(1, 2, \dots, j-1; j, j+1, \dots, n) = (-1)^{j-1} \sum_{\sigma \in \text{COP}\{\alpha\}\{\beta\}} A_{n;1}^{\text{adj}}(\sigma(1, 2, \dots, n)), \quad (\text{I.1})$$

where $\alpha_i \in \{\alpha\} \equiv \{j-1, j-2, \dots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{j, j+1, \dots, n-1, n\}$, and the set of permutations $\text{COP}\{\alpha\}\{\beta\}$ is defined below eqn. (4.4). We have added the superscript “adj” to (I.1) to avoid confusion with the $\bar{q}qg \dots g$ amplitudes denoted by $A_{n;j}$.

In ref. [20] a string-theory based proof of eqn. (I.1) was presented. We now review this proof, but in terms of the color-ordered Feynman rules in fig. 1. The leading-color partial amplitude $A_{n;1}^{\text{adj}}(1, 2, \dots, n)$ associated with $\text{Tr}(T^{a_1}T^{a_2} \dots T^{a_n})$ is given by the sum of all color-ordered planar diagrams whose external legs follow the cyclic ordering of the color trace, as depicted in fig. 2. Examples of five-point diagrams, dressed with color flow lines, are shown in fig. 6. Our convention for the direction of the color flow is to follow the reverse ordering of the color trace; this will lead to the standard convention for the color arrow following the fermion arrow after conversion to fundamental-representation fermions. To compute the subleading-color amplitudes $A_{n;j}^{\text{adj}}$ associated with the color structure $\text{Tr}(T^{a_1}T^{a_2} \dots T^{a_{j-1}}) \times \text{Tr}(T^{a_j}T^{a_{j+1}} \dots T^{a_n})$ directly, one must sum over all planar color-ordered diagrams whose corresponding Feynman diagrams can give rise to this color structure. These are the diagrams where the cyclic ordering of legs that belong to each trace follows the ordering of that trace, but where the ordering in one trace is reversed because the two color lines associated with a adjoint particle flow in opposite directions around the loop. For example, in fig. 7 the legs follow ordering 123456, but the color structure is $\text{Tr}(T^{a_4}T^{a_3}T^{a_2}) \times \text{Tr}(T^{a_5}T^{a_6}T^{a_1})$.

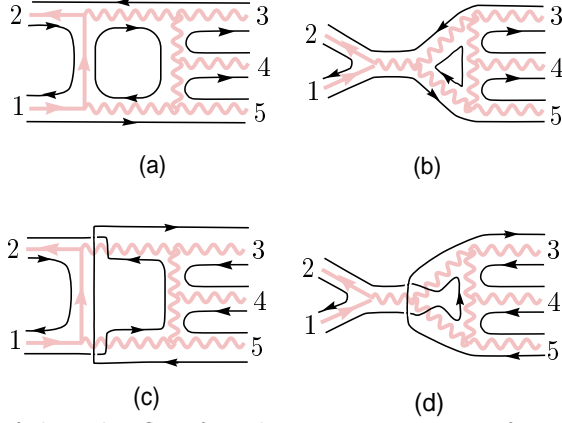


Figure 6. Examples of the color flow for adjoint representation fermions, using the double-line formalism. The Feynman diagrams are gray, and the oriented color lines dressing them are black.

Thus we sum over color-ordered diagrams with the legs permuted over $COP\{\alpha\}\{\beta\}$, where $\alpha = \{j-1, j-2, \dots, 1\}$, $\beta = \{j, j+1, \dots, n\}$. We must, however, explicitly exclude one class of diagrams whose color flow is incorrect. This is the class of diagrams where indices from *both* sets $\{\alpha\}$ and $\{\beta\}$ label leaves of the same tree attached to the loop. The color flow in these diagrams cannot produce the desired trace structure, because the line attaching the tree to the loop can carry only a single pair of color indices, whereas two pairs would be required to join both α and β indices to those elsewhere on the loop. Examples of diagrams contributing to the color structure $\text{Tr}(T^{a_1}T^{a_2}) \times \text{Tr}(T^{a_5}T^{a_4}T^{a_3})$ are shown in figs. 6c and 6d, and one should exclude diagrams such as the one depicted in fig. 8. (In the string-based derivation [20], another class of diagrams — those where a single tree contains *all* elements of either $\{\alpha\}$ or $\{\beta\}$ — is also excluded explicitly. This class automatically cancels out of the field-theory calculation, so long as the set $\{\alpha\}$ or $\{\beta\}$ in question contains only legs transforming under the adjoint representation.)

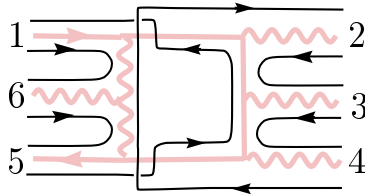


Figure 7. The ordering of one trace is reversed as compared to the other.

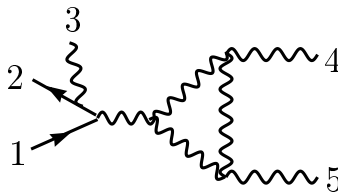


Figure 8. An example of a diagram that does not contribute to the coefficient of the color structure $\text{Tr}(T^{a_1}T^{a_2}) \times \text{Tr}(T^{a_5}T^{a_4}T^{a_3})$.

One can divide the set of all diagrams into a ‘parent’ subset, which have only three-point vertices and no non-trivial trees (depicted in fig. 5), and all the remaining diagrams. We refer to the latter as ‘daughter’ diagrams, because each can be derived from some ‘parent’ diagram via a continuous ‘pinching’ process, in which two lines attached to the loop are brought together to a four-point interaction, or further pulled out from the loop, and left as the branches of a tree attached to the loop. Repeating this process in all inequivalent ways yields all graphs contributing to the same color ordering as the parent diagram, i.e. all of its daughter diagrams. For instance, the color-ordered diagram in fig. 8 is a daughter of the parent with ordering 12345 depicted in fig. 6a. Daughter diagrams of different parents can be essentially the same diagram. For example, if one swaps legs 2 and 3 in fig. 8, one obtains a daughter of the parent with ordering 13245, but the two daughter diagrams differ only by an overall minus sign coming from the antisymmetric color-ordered three-vertex. Such relations are important for proving eqns. (I.1) and (4.4).

Let us first focus on the ‘parent’ subset of diagrams. As mentioned in section 4, by performing a color decomposition of ordinary Feynman diagrams and using eqns. (2.1) and (2.2) one can show that all ‘parent’ diagrams feed into both $A_{n;1}^{\text{adj}}$ and $A_{n;j>1}^{\text{adj}}$ in the correct way so that eqn. (I.1) is satisfied for this class of diagrams. The same arguments apply to those ‘daughter’ diagrams where each pinched-off tree contains only members of the $\{\alpha\}$ set, or only members of the $\{\beta\}$ set.

The only thing left to prove is that the class of daughter diagrams specifically excluded from $A_{n;j}$ — where individual trees have both $\{\alpha\}$ and $\{\beta\}$ members — does yield a vanishing contribution when summed over the permutations in $COP\{\alpha\}\{\beta\}$. Eqn. (I.1) then follows. When such diagrams have only three-point vertices on the trees, the diagrams can be arranged so they cancel in pairs. The pairs are related by the exchange of an $\{\alpha\}$ leg with a $\{\beta\}$ leg on a tree; the cancellation follows from the anti-symmetry of the three-point color-ordered Feynman vertices in fig. 1 under the interchange of the ordering of the two outer legs. For example, the pairs of diagrams in fig. 9 cancel in the sum. For diagrams with trees containing four-point vertices, using the color-ordered rules in fig. 1, the cancellations in the sum over $COP\{\alpha\}\{\beta\}$ occur in triplets, such as those shown in fig. 10. Diagrams with four-point vertices attached to the loop can be decomposed into the same color structures encountered above. This provides a purely field-theoretic proof of eqn. (I.1),

where all external and internal states are in the adjoint representation, independent of whether they are gluons or adjoint fermions; in particular it applies to the subleading-color $\tilde{g}\tilde{g}\dots g$ super-Yang-Mills partial amplitudes.

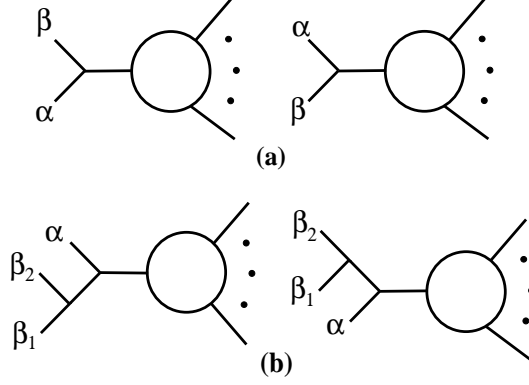


Figure 9. Examples of pairs of diagrams that cancel in the permutation sum.

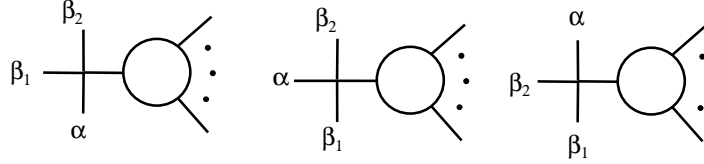


Figure 10. Diagrams with four-point vertices cancel in the permutation sum in triplets.

Now consider the modifications necessary for $\bar{q}qg\dots g$ QCD amplitudes, where the two fermions are fundamental representation quarks instead of adjoint representation gluinos. We will show that the subleading-color partial amplitudes $A_{n;j>2}(1_{\bar{q}}, 2_q; 3, \dots, n)$ in the full amplitude (2.4), omitting for now the $n_{f,s}$ -dependent terms from closed fermion or scalar loops, are given by exactly the same type of formula as eqn. (I.1),

$$A_{n;j}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, j+3, \dots, n)|_{\text{no } n_{f,s}} = (-1)^{j-1} \sum_{\sigma \in COP\{\alpha\}\{\beta\}} A_n^{L,[1]}(\sigma(1_{\bar{q}}, 2_q, 3, \dots, n)), \quad (\text{I.2})$$

where $\alpha_i \in \{\alpha\} \equiv \{j+1, j, \dots, 4, 3\}$, $\beta_i \in \{\beta\} \equiv \{1, 2, j+2, j+3, \dots, n-1, n\}$, $COP\{\alpha\}\{\beta\}$ is defined in exactly the same way as for the adjoint representation formula, and $n_{f,s}$ means either n_s or n_f .

The main difference between the adjoint formula (I.1) and the fundamental formula (I.2) is that the adjoint formula lumps ‘left’ diagrams, where the fermion goes around the loop on the left

side, together with otherwise identical ‘right’ diagrams, whereas the fundamental formula keeps the two separate, as required by their different color flows. To derive (I.2) from (I.1) we remove a single color line from a special subset of the $\tilde{g}\tilde{g}g\dots g$ partial amplitudes $A_{n;j}^{\text{adj}}$ on the left-hand side of (I.1), those where the two gluino charge matrices are in the same trace and are adjacent to each other; this subset is in one-to-one correspondence with the $\bar{q}qg\dots g$ partial amplitudes $A_{n;j}$. We show that this color-line removal corresponds to dropping the unwanted ‘right’ set of diagrams on the right-hand side of (I.1), thus converting $A_{n;1}^{\text{adj}}$ to A_n^L . As discussed above, the $-1/N_c$ Fierz corrections can be ignored here.

Let us focus on the coefficient of the same-trace color structure $\text{Gr}_{n;j}^s$ (for some fixed j) in the modified color decomposition (2.8),

$$\begin{aligned} \text{Tr}(T^{a_3} \dots T^{a_{j+1}}) \text{Tr}(T^{f_1} T^{f_2} T^{a_{j+2}} \dots T^{a_n}), & \quad j = 3, \dots, n-2, \\ \text{Tr}(T^{a_3} \dots T^{a_n}) \text{Tr}(T^{f_1} T^{f_2}), & \quad j = n-1, \end{aligned} \quad (\text{I.3})$$

whose corresponding partial amplitude is $A_{n;j}^{\text{adj}}(3, \dots, j+1; 1_{\tilde{g}}, 2_{\tilde{g}}, j+2, j+3, \dots, n)$. Our convention for drawing the color-dressed diagrams is to impose a clockwise ordering on the legs associated with the color trace containing the fermion color matrices and a counterclockwise ordering on the legs associated with the other trace. In every double-line diagram contributing to this partial amplitude, there is a color line that runs directly from gluino 2 to gluino 1 along the gluino line, as examples in fig. 6 illustrate. If we remove this color line (as shown in fig. 11), then (I.3) is converted to

$$\begin{aligned} \text{Gr}_{n;j}^{(\bar{q}q)}(3, \dots, j+1; j+2, \dots, n) &= \text{Tr}(T^{a_3} \dots T^{a_{j+1}}) (T^{a_{j+2}} \dots T^{a_n})_{i_2}^{\bar{i}_1}, \quad j = 3, \dots, n-2, \\ \text{Gr}_{n;n-1}^{(\bar{q}q)}(3, \dots, n) &= \text{Tr}(T^{a_3} \dots T^{a_n}) \delta_{i_2}^{\bar{i}_1}, \end{aligned} \quad (\text{I.4})$$

that is to the fundamental representation color structures given in eqn. (2.5), with coefficients $A_{n;j}$.

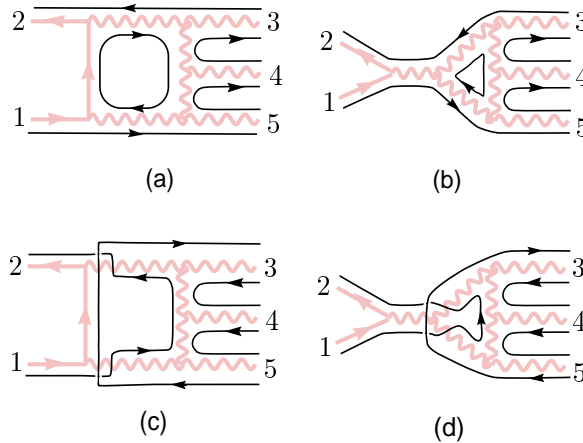


Figure 11. Examples of color stripping; the color lines running directly from gluino 2 to gluino 1 in fig. 6 have been removed.

A given $\tilde{g}\tilde{g}g \dots g$ diagram, with one color line running between the two gluinos removed, can be interpreted as a $\bar{q}qg \dots g$ diagram, but only if the color line to be removed runs along the fermion side of the loop. This requirement eliminates half the diagrams, namely the ‘right’ diagrams contributing to A_n^R , leaving precisely the desired ‘left’ diagrams contributing to A_n^L . As a particular example, the two routings of the gluino through the diagrams depicted in fig. 12a,b contribute to the color structure (I.3). But in fig. 12b the color line running directly between the two gluinos must run along the gluon side of the loop in order to generate (I.3); therefore this diagram should be dropped in converting to $\bar{q}qg \dots g$, while fig. 12a should be kept. This shows that replacing $A_{n;1}$ by A_n^L on the right-hand side of eqn. (I.1) is the correct prescription for the ‘parent’ diagrams. However, we must again show that all unwanted diagrams with attached trees cancel in the permutation sum. As was the case for the adjoint representation case, in each term on the right-hand side of eqn. (I.2) we are including diagrams which do not belong on the left-hand side, because the color flow they represent does not allow them to contribute. The argument is similar to the one for the adjoint case. If diagrams contain trees with leaves labeled by indices from both sets $\{\alpha\}$ and $\{\beta\}$, then such diagrams cancel in the permutation sum exactly as for the adjoint case. These cancellations are due to the antisymmetry of the color-stripped vertices, and are thus independent of the color representation of the fermions. Note that ‘left’ diagrams cancel against ‘left’, and ‘right’ against ‘right’. This completes the conversion of the adjoint formula (I.1) to the fundamental formula (I.2), excluding the contributions of closed fermion or scalar loops.

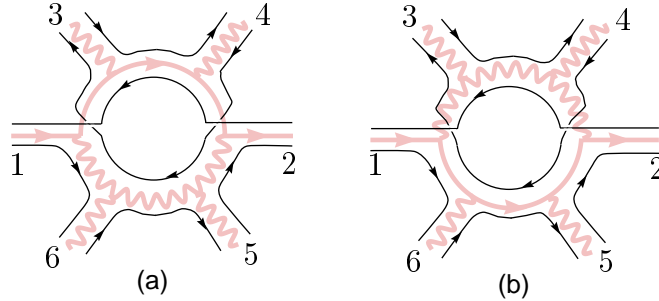


Figure 12. Both ‘left’ (a) and ‘right’ (b) diagrams contribute to adjoint representation gluino partial amplitudes, but only the ‘left’ diagram contributes for fundamental representation quarks.

We turn next to the contributions of closed fermion or scalar loops in the fundamental representation. In this case, neglecting the $-1/N_c$ term in the $SU(N_c)$ Fierz identity (2.2) leads only to the $n_{f,s}$ -dependent terms in $A_{n;1}$ in eqn. (4.1), and does not yield a contribution to $A_{n;j>1}$. Including the $-1/N_c$, or $U(1)$ subtraction, term in the gluon propagator connecting the external $\bar{q}q$ line to the fermion/scalar loop decouples the color flow for the tree containing $\bar{q}q$ from the loop

color flow. However, the color-stripping argument can still be applied to these terms, if we allow the stripped color line to propagate down the ‘photon’, around the loop, and back again along the ‘photon’. We start with the $\tilde{g}\tilde{g}\dots g$ double-line configuration where a color line starts at gluino 2, flows down the gauge boson line connecting the external fermion legs to an adjoint fermion (or scalar) loop, flows around the loop and then returns through the gauge boson line to wind up at gluino 1, as shown in fig. 13a. We remove this color line, as shown in fig. 13b, to obtain the $U(1)$ subtraction contribution for $\bar{q}qg\dots g$. In this case it is the ‘right’ type diagrams that have the correct color flow to be strippable, so the appropriate formula (including the Fierz factor of $-1/N_c$) is

$$A_{n;j}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, j+3, \dots, n)|_{n_{f,s} \text{ terms}} = (-1)^{j-1} \sum_{\sigma \in COP\{\alpha\}\{\beta\}} \left[-\frac{n_f}{N_c} A_n^{R,[1/2]}(\sigma(1_{\bar{q}}, 2_q, 3, \dots, n)) - \frac{n_s}{N_c} A_n^{R,[0]}(\sigma(1_{\bar{q}}, 2_q, 3, \dots, n)) \right]. \quad (\text{I.5})$$

The cancellation of unwanted diagrams with attached trees from the right-hand side of eqn. (I.5) works as in the no- $n_{f,s}$ case.

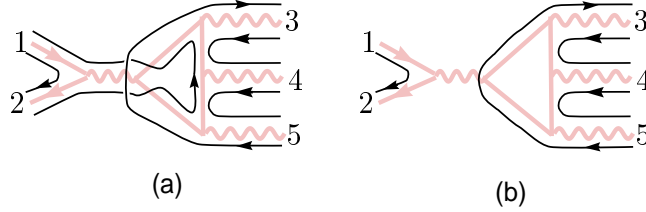


Figure 13. Color stripping the gluon line connecting the external fermion line to the fermion loop in (a) produces the ‘photon’ subtraction diagram in (b).

Thus the final formula for subleading-color $\bar{q}qg\dots g$ partial amplitudes in terms of primitive amplitudes is the one given in eqn. (4.4).

Appendix II. Four-Point Amplitudes

In this appendix, we collect the four-gluon and two-quark two-gluon amplitudes, needed for checking the collinear limits of the two-quark three-gluon amplitudes. These amplitudes agree with the results of KST [16]. Note, however, that these authors used a different color decomposition and overall sign convention than used in this paper.

We begin by listing all tree amplitudes that appear in the collinear limits, eqn. (III.1), of the

five-point amplitudes presented in this paper,

$$\begin{aligned}
A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) &= i \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle}, \\
A_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+) &= i \frac{\langle 1 3 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle}, \\
A_4^{\text{tree}}(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+) &= i \frac{\langle 1 3 \rangle^3 \langle 2 3 \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle}, \\
A_4^{\text{tree}}(1_{\bar{q}}^-, 2_q^+, 3^+, 4^-) &= i \frac{\langle 1 4 \rangle^3 \langle 2 4 \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle}, \\
A_4^{\text{tree}}(1_{\bar{q}}^-, 2^-, 3_q^+, 4^+) &= i \frac{\langle 1 2 \rangle^3 \langle 3 2 \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle}.
\end{aligned} \tag{II.1}$$

Now consider the one-loop four-point amplitudes, beginning with the four-gluon amplitudes. Amplitudes with only external gluons may be decomposed in terms of contributions to supersymmetric multiplets

$$A_{n;1}(1, 2, \dots, n) = A_n^g + \left(4 - \frac{n_f}{N_c}\right) A_n^f + \left(1 + \frac{n_s}{N_c} - \frac{n_f}{N_c}\right) A_n^s, \tag{II.2}$$

where A^g is the contribution of an $N = 4$ multiplet, $-A^f$ the contribution of an $N = 1$ chiral multiplet, and A^s the contribution of a complex scalar.

The finite four-gluon amplitudes are

$$\begin{aligned}
A_4^g(1^+, 2^+, 3^+, 4^+) &= 0, \\
A_4^f(1^+, 2^+, 3^+, 4^+) &= 0, \\
A_4^s(1^+, 2^+, 3^+, 4^+) &= \frac{i}{48\pi^2} \frac{s_{12}s_{23}}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle}, \\
A_4^g(1^-, 2^+, 3^+, 4^+) &= 0, \\
A_4^f(1^-, 2^+, 3^+, 4^+) &= 0, \\
A_4^s(1^-, 2^+, 3^+, 4^+) &= \frac{i}{48\pi^2} \frac{\langle 2 4 \rangle [2 4]^3}{[1 2] \langle 2 3 \rangle \langle 3 4 \rangle [4 1]}.
\end{aligned} \tag{II.3}$$

The tree amplitudes vanish for these helicity configurations.

For the amplitudes containing infrared and ultraviolet singularities, we further decompose the amplitude into V and F pieces of the sort given in eqn. (5.6)

$$A_4^x = c_{\Gamma}(V^x A_4^{\text{tree}} + iF^x), \quad x = g, f, s. \tag{II.4}$$

The universal functions

$$V_{4,\text{gluon}}^g = -\frac{2}{\epsilon^2} \left[\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon \right] + \ln^2 \left(\frac{-s_{12}}{-s_{23}} \right) + \pi^2, \tag{II.5}$$

appear in the four-point amplitudes with infrared divergences.

For the four-gluon amplitude $A_{4;1}(1^-, 2^-, 3^+, 4^+)$ we have

$$\begin{aligned} V^g &= V_{4,\text{gluon}}^g, & F^g &= 0, \\ V^f &= -\frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon - 2, & F^f &= 0, \\ V^s &= -\frac{V^f}{3} + \frac{2}{9}, & F^s &= 0. \end{aligned} \tag{II.6}$$

For $A_{4;1}(1^-, 2^+, 3^-, 4^+)$ we have

$$\begin{aligned} V^g &= V_{4,\text{gluon}}^g, & F^g &= 0, \\ V^f &= -\frac{1}{2\epsilon} \left(\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon \right) - 2, \\ V^s &= -\frac{V^f}{3} + \frac{2}{9}, \\ F^f &= \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \left[-\frac{s_{12}s_{23}}{2s_{13}^2} \left(\ln^2 \left(\frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) + \frac{1}{2} \frac{s_{12} - s_{23}}{s_{13}} \ln \left(\frac{-s_{12}}{-s_{23}} \right) \right], \\ F^s &= \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \left(-\frac{s_{12}s_{23}}{s_{13}^2} \right) \left[-\frac{s_{12}s_{23}}{s_{13}^2} \left(\ln^2 \left(\frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) \right. \\ &\quad \left. + \frac{s_{12} - s_{23}}{s_{13}} \left(1 + \frac{s_{13}^2}{6s_{12}s_{23}} \right) \ln \left(\frac{-s_{12}}{-s_{23}} \right) + 1 \right]. \end{aligned} \tag{II.7}$$

For the finite amplitudes with two quarks and two gluons we have

$$\begin{aligned} A_4^{\text{SUSY}}(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+) &= 0, \\ A_4^L(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+) &= -\frac{i}{32\pi^2} \frac{\langle 12 \rangle [24]}{\langle 23 \rangle \langle 34 \rangle} + A_4^s(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+), \\ A_4^s(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+) &= -\frac{i}{48\pi^2} \frac{s_{23}}{s_{12}} \frac{\langle 12 \rangle [24]}{\langle 23 \rangle \langle 34 \rangle}, \\ A_4^f(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+) &= 0. \end{aligned} \tag{II.8}$$

$$\begin{aligned} A_4^{\text{SUSY}}(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+) &= 0, \\ A_4^L(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+) &= -\frac{i}{32\pi^2} \frac{\langle 13 \rangle [24]}{\langle 23 \rangle \langle 34 \rangle}, \\ A_4^s(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+) &= 0, \\ A_4^f(1_{\bar{q}}^-, 2^+, 3_q^+, 4^+) &= 0. \end{aligned} \tag{II.9}$$

For the helicities containing infrared and ultraviolet singularities we decompose the amplitude as in the five-point case (eqn. (5.6)), and again we have $V^s = V^f = 0$. For $A_4(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+)$ we have

$$\begin{aligned} V^{\text{SUSY}} &= V_{4,\text{gluon}}^g - \frac{3}{2\epsilon} \left(\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon \right) - 6, \\ V^L &= \left[V_{4,\text{gluon}}^g + \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon \right] - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{5}{2}, \end{aligned} \tag{II.10}$$

$$\begin{aligned}
F^{\text{SUSY}} &= 3 \frac{\langle 13 \rangle^3 \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \left[-\frac{s_{12}s_{23}}{2s_{13}^2} \left(\ln^2 \left(\frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) + \frac{1}{2} \frac{s_{12} - s_{23}}{s_{13}} \ln \left(\frac{-s_{12}}{-s_{23}} \right) \right], \\
F^L &= \frac{\langle 13 \rangle^3 \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{s_{12}}{2s_{13}} \left[\left(3 - \frac{2s_{23}}{s_{13}} \right) \ln \left(\frac{-s_{12}}{-s_{23}} \right) + \left(\frac{s_{12}^2}{s_{13}^2} - \frac{3s_{23}}{s_{13}} \right) \left(\ln^2 \left(\frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) + 1 \right], \\
F^s &= 0, \\
F^f &= 0.
\end{aligned} \tag{II.11}$$

For $A_4(1_{\bar{q}}^-, 2_q^+, 3^+, 4^-)$ we have

$$\begin{aligned}
V^{\text{SUSY}} &= V_{4,\text{gluon}}^g - \frac{3}{\epsilon} \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon - 6, \\
V^L &= \left[V_{4,\text{gluon}}^g + \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon \right] - \frac{3}{2\epsilon} \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{5}{2},
\end{aligned} \tag{II.12}$$

$$\begin{aligned}
F^{\text{SUSY}} &= 0, \\
F^L &= \frac{\langle 14 \rangle^3 \langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{s_{12}}{2s_{13}} \left[\ln^2 \left(\frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right], \\
F^s &= 0, \\
F^f &= 0.
\end{aligned} \tag{II.13}$$

For $A_4(1_{\bar{q}}^-, 2^-, 3_q^+, 4^+)$ we have

$$\begin{aligned}
V^{\text{SUSY}} &= V_{4,\text{gluon}}^g - \frac{3}{\epsilon} \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon - 6, \\
V^L &= \frac{1}{2} V^{\text{SUSY}},
\end{aligned} \tag{II.14}$$

$$\begin{aligned}
F^{\text{SUSY}} &= 0, & F^L &= 0, \\
F^s &= 0, & F^f &= 0.
\end{aligned} \tag{II.15}$$

The remaining amplitude, $A_4(1_{\bar{q}}^-, 2^+, 3_q^+, 4^-)$, may be obtained via a reflection,

$$A_4(1_{\bar{q}}^-, 2^+, 3_q^+, 4^-) = A_4(1_{\bar{q}}^-, 4^-, 3_q^+, 2^+). \tag{II.16}$$

Appendix III. Collinear Limit Checks

In this appendix, we illustrate the use of collinear limits in verifying the correctness of explicitly calculated amplitudes, including signs and normalizations. As the momenta of two color-adjacent external legs become collinear, the amplitudes factorize into sums of lower point amplitudes multiplied by ‘splitting amplitudes’. The constraints provided by the collinear limits are sufficiently restrictive that they can be used to construct ansätze for higher-point amplitudes based on known

lower point amplitudes [23,20]. The collinear-limit checks may also be used in numerical programs, when converting the amplitudes presented in this paper to physical cross-sections.

At one loop, the collinear limits of color-ordered one-loop QCD amplitudes have the form [23,20]

$$A_n^{\text{loop}} \xrightarrow{a\parallel b} \sum_{\lambda=\pm} \left(\text{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{loop}}(\dots (a+b)^\lambda \dots) \right. \\ \left. + \text{Split}_{-\lambda}^{\text{loop}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}(\dots (a+b)^\lambda \dots) \right), \quad (\text{III.1})$$

where $k_a \rightarrow zP$ and $k_b \rightarrow (1-z)P$, with $P = k_a + k_b$, $P^2 = s_{ab} \rightarrow 0$. The tree and loop splitting amplitudes, $\text{Split}_{-\lambda}^{\text{tree}}$ and $\text{Split}_{-\lambda}^{\text{loop}}$, behave as $1/\sqrt{s_{ab}}$ in this limit. This formula holds for any of the primitive amplitudes — which at loop level may carry the additional labels $J = 1, 1/2, 0$ (for n -gluon amplitudes), or $x = \text{SUSY}, L, s, f$ (for two-quark $(n-2)$ -gluon amplitudes) — as well as for the leading-color amplitudes $A_{n;1}$, provided that one assigns the correct labels to all ‘loop’ quantities in (III.1). A tabulation of the splitting amplitudes appearing in massless QCD computations was given in appendix B of ref. [20]. The results were given in terms of leading-color partial amplitudes $A_{n;1}$ rather than primitive amplitudes; below we shall convert the results to the latter form. We describe the collinear behavior of amplitudes before subtraction of the ultraviolet pole. A proof of the universality of the splitting amplitudes, limited to external gluons and scalar loops, was outlined in refs. [23,39]; a more complete treatment will be given elsewhere. The power of the collinear-limit constraint arises from the fact that relationship (III.1) must hold in every channel.

In order to use the splitting amplitudes in appendix B of ref. [20] for the primitive amplitudes presented here, we must take into account that the overall sign convention used in ref. [20] for the helicity amplitudes $A(1_{\bar{q}}^-, 2_q^+, 3, \dots, n)$ is opposite to that used for primitive amplitudes in the present paper. (The overall sign convention for $A(1_q^+, 2_{\bar{q}}^-, 3, \dots, n)$ is the same.) With the former sign convention, one cannot relabel $\bar{q} \rightarrow q$, $q \rightarrow \bar{q}$ in amplitudes without introducing extra signs. Equivalently, one cannot directly interpret the tree-level quark amplitudes as gluino amplitudes, because the extra constraints on gluino amplitudes imposed by the Majorana nature of the gluino are not satisfied with the former choice of sign. The change in amplitude conventions implies a sign change in the second and fourth tree splitting amplitudes in eqn. (B.5) of ref. [20]. With these sign changes the tree splitting amplitudes in that appendix may be used for the primitive amplitudes presented in this paper; furthermore, one may relabel $\bar{q} \rightarrow q$, $q \rightarrow \bar{q}$ in the splitting amplitudes to get the additional orderings $\text{Split}_{-\lambda}(\bar{q}, g)$ and $\text{Split}_{-\lambda}(g, q)$ which appear in limits of the primitive amplitudes.

For all splitting amplitudes which do not vanish at tree level, the proportionality constant r_S is defined by

$$\text{Split}_{-\lambda}^{\text{loop}}(a^{\lambda_a}, b^{\lambda_b}) = c_\Gamma \times \text{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) \times r_S(-\lambda, a^{\lambda_a}, b^{\lambda_b}) . \quad (\text{III.2})$$

For any collinear limit of a supersymmetric primitive amplitude A_n^{SUSY} , r_S is given by the function r_S^{SUSY} given in eqn. (B.13) of ref. [20].

For the non-supersymmetric primitive amplitudes ($x = L, s, f$) we first discuss the case $g \rightarrow gg$. Because the fermion line is routed on a definite side of the loop in these components, the loop splitting amplitudes depend on whether the pair of adjacent collinear gluons a, b are between \bar{q} and q ($A_n^x(1_{\bar{q}}, \dots, a, b, \dots, 2_q, \dots, n)$) or between q and \bar{q} ($A_n^x(1_{\bar{q}}, \dots, 2_q, \dots, a, b, \dots, n)$). In the first case the $x = L, s, f$ loop splitting amplitudes all vanish. In the second case they are given by the pure-gluon ($J = 1$) contribution for $x = L$, the negative of the sum of the $J = 1/2$ and $J = 0$ contributions for $x = f$, and the $J = 0$ contribution for $x = s$:

$$\text{Split}_+^x(a^+, b^+) = \begin{cases} 0, & x = \text{SUSY}, f ; \\ \text{Split}_+^{[1]}(a^+, b^+), & x = L, s \end{cases} \quad (\text{III.3})$$

(this is the one case where the tree splitting amplitude vanishes), and

$$\begin{aligned} r_S^L(\pm, a, b) &= r_S^{[1]}(\pm, a, b) , \\ r_S^f(\pm, a, b) &= 0 , \\ r_S^s(\pm, a, b) &= r_S^{[0]}(\pm, a, b) . \end{aligned} \quad (\text{III.4})$$

The quantities $\text{Split}_+^{[1]}(a^+, b^+)$ and $r_S^{[J]}$ are given in eqns. (B.8) and (B.9) of ref. [20].

The r_S functions for the non-supersymmetric $g \rightarrow \bar{q}q$ splitting amplitudes are

$$\begin{aligned} r_S^L(\pm, \bar{q}^\mp, q^\pm) &= -\frac{1}{\epsilon^2} \left[\left(\frac{\mu^2}{z(1-z)(-s_{\bar{q}q})} \right)^\epsilon - \left(\frac{\mu^2}{(-s_{\bar{q}q})} \right)^\epsilon \right] \\ &\quad + \frac{13}{6\epsilon} \left(\frac{\mu^2}{(-s_{\bar{q}q})} \right)^\epsilon + 2 \ln(z) \ln(1-z) - \frac{\pi^2}{6} + \frac{83}{18} , \\ r_S^s(\pm, \bar{q}^\mp, q^\pm) &= -\frac{1}{3\epsilon} \left(\frac{\mu^2}{(-s_{\bar{q}q})} \right)^\epsilon - \frac{8}{9} , \\ r_S^f(\pm, \bar{q}^\mp, q^\pm) &= \frac{1}{\epsilon} \left(\frac{\mu^2}{(-s_{\bar{q}q})} \right)^\epsilon + 2 . \end{aligned} \quad (\text{III.5})$$

For the non-supersymmetric $q \rightarrow qq$ splitting amplitudes, one obtains different results depending on whether the gluon (denoted by a) is before or after the quark, with respect to the clockwise ordering of the vertices. The difference is again due to the routing of the fermion line in

the primitive amplitudes. We have

$$\begin{aligned}
r_S^L(q^-, a^+) &= r_S^L(q^+, a^-) = f^L(1-z, s_{qa}) , \\
r_S^L(q^-, a^-) &= r_S^L(q^+, a^+) = f^L(1-z, s_{qa}) + \frac{1-z}{2} , \\
r_S^L(a^+, \bar{q}^+) &= r_S^L(a^-, \bar{q}^-) = f^L(z, s_{a\bar{q}}) + \frac{z}{2} , \\
r_S^L(a^-, \bar{q}^+) &= r_S^L(a^+, \bar{q}^-) = f^L(z, s_{a\bar{q}}) , \\
r_S^L(a^+, q^-) &= r_S^L(a^-, q^+) = f^R(z, s_{aq}) , \\
r_S^L(a^-, q^-) &= r_S^L(a^+, q^+) = f^R(z, s_{aq}) - \frac{z}{2} , \\
r_S^L(\bar{q}^+, a^+) &= r_S^L(\bar{q}^-, a^-) = f^R(1-z, s_{\bar{q}a}) - \frac{1-z}{2} , \\
r_S^L(\bar{q}^+, a^-) &= r_S^L(\bar{q}^-, a^+) = f^R(1-z, s_{\bar{q}a}) ,
\end{aligned} \tag{III.6}$$

where the functions f^L and f^R correspond to the leading- and subleading-color parts of the function f defined in eqn. (B.11) of ref. [20],

$$\begin{aligned}
f^L(z, s) &= -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{z(-s)} \right)^\epsilon - \text{Li}_2(1-z) , \\
f^R(z, s) &= -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{(1-z)(-s)} \right)^\epsilon + \frac{1}{\epsilon^2} \left(\frac{\mu^2}{(-s)} \right)^\epsilon - \text{Li}_2(z) , \\
f(z, s) &= f^L(z, s) - \frac{1}{N_c^2} f^R(z, s) .
\end{aligned} \tag{III.7}$$

The $x = s, f$ parts of the $q \rightarrow qg$ splitting amplitudes all vanish.

As an illustration of the collinear limits consider the amplitude $A_5^f(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+, 5^+)$ given by

$$A_5^f(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+, 5^+) \equiv i_{c_\Gamma} F^f = -i_{c_\Gamma} \frac{\langle 13 \rangle^2 [45]}{\langle 12 \rangle \langle 45 \rangle} \frac{\ln\left(\frac{-s_{12}}{-s_{34}}\right)}{s_{34} - s_{12}} . \tag{III.8}$$

The collinear limits of A_5^f may be considered independently of all other primitive amplitudes because they are separately gauge invariant; in QCD amplitudes A_5^f enters with a coefficient proportional to the number of fermions n_f , which distinguishes it from the other primitive amplitudes. (Although A_5^s also enters QCD amplitudes with coefficients containing n_f , it does so only in the combination $n_s - n_f$ and it may therefore be treated independently.)

First consider the limit as gluon 3 becomes collinear with gluon 4 ($3 \parallel 4$), and also the $4 \parallel 5$ collinear limit. In either of these cases, eqn. (III.8) does not contain a collinear singularity. Compare this result to the expectation from eqn. (III.1). These limits are particularly simple to analyze because the A_4^f amplitudes with two quarks and two gluons vanish for all helicities, $A_4^f(1_{\bar{q}}, 2_q, 3, 4) = 0$ (see eqns. (II.8), (II.11) and (II.13)). Additionally, the loop splitting amplitudes vanish for all helicities, $\text{Split}_\pm^f(a^\pm, b^\pm) = 0$, where $\text{Split}_\pm^f \equiv -(\text{Split}_\pm^{[1/2]} + \text{Split}_\pm^{[0]})$, and $\text{Split}_\pm^{[1/2]}$,

$\text{Split}_{\pm}^{[0]}$ are defined in appendix B of ref. [20]. Thus, all contributions to the right-hand side of eqn. (III.1) vanish for the $3 \parallel 4$ and $4 \parallel 5$ collinear limits, in agreement with the lack of collinear singularities in these channels in the amplitude (III.8).

The $1_{\bar{q}} \parallel 2_q$ collinear limit is a bit less trivial since the expression (III.8) does contain a collinear pole. Using the explicit value of A_5^f in eqn. (III.8) we have

$$A_5^f(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+, 5^+) \xrightarrow{1 \parallel 2} -i c_{\Gamma} \frac{z}{\langle 1 2 \rangle} \frac{\langle P 3 \rangle^2 [4 5]}{\langle 4 5 \rangle} \frac{\ln\left(\frac{-P^2}{-s_{34}}\right)}{s_{34}}, \quad (\text{III.9})$$

where we have taken $k_1 = zP$ and $k_2 = (1 - z)P$. Now compare this to the result obtained from the collinear formula (III.1). The necessary four-gluon amplitudes are

$$\begin{aligned} A_4^{\text{tree}}(P^+, 3^-, 4^+, 5^+) &= 0, \\ A_4^{\text{tree}}(P^-, 3^-, 4^+, 5^+) &= i \frac{\langle P 3 \rangle^4}{\langle P 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 P \rangle}, \\ A_4^f(P^+, 3^-, 4^+, 5^+) &= 0, \\ A_4^f(P^-, 3^-, 4^+, 5^+) &= -c_{\Gamma} \left(\frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{34}} \right)^{\epsilon} + 2 \right) A_4^{\text{tree}}(P^-, 3^-, 4^+, 5^+), \end{aligned} \quad (\text{III.10})$$

while the relevant splitting functions are

$$\begin{aligned} \text{Split}_+^{\text{tree}}(\bar{q}^-, q^+) &= \frac{z}{\langle \bar{q} q \rangle}, \\ \text{Split}_+^f(\bar{q}^-, q^+) &= c_{\Gamma} \left(\frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{\bar{q}q}} \right)^{\epsilon} + 2 \right) \frac{z}{\langle \bar{q} q \rangle}. \end{aligned} \quad (\text{III.11})$$

Plugging these results into eqn. (III.1) we obtain for the non-vanishing terms

$$\begin{aligned} A_5^f(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+, 5^+) &\xrightarrow{1 \parallel 2} \text{Split}_+^{\text{tree}}(1_{\bar{q}}^-, 2_q^+) A_4^f(P^-, 3^-, 4^+, 5^+) + \text{Split}_+^f(1_{\bar{q}}^-, 2_q^+) A_4^{\text{tree}}(P^-, 3^-, 4^+, 5^+) \\ &= -i c_{\Gamma} \frac{z}{\langle 1 2 \rangle} \ln\left(\frac{-P^2}{-s_{34}}\right) \frac{\langle P 3 \rangle^4}{\langle P 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 P \rangle} \end{aligned} \quad (\text{III.12})$$

which reproduces eqn. (III.9), after spinor helicity simplifications.

One can continue in this way, systematically verifying that the collinear limits in all channels are correct. We have done so for all amplitudes presented in this paper.

In checking the collinear limits, the behavior of most of the functions appearing in the amplitudes is straightforward to obtain. We present here only the one function with a slightly complicated limit. We take the collinear limit of two color-adjacent momenta k_c and k_{c+1} ; denoting the sum by P , the momentum fraction z satisfies $k_c = zP$ and $k_{c+1} = (1 - z)P$ in the limit. Furthermore, we

will denote the limit of $s_{c+2,c-2}$ by s and the limit of $s_{c-2,c-1}$ by t . With these definitions we have

$$\begin{aligned}
& \frac{\text{LS}_2 \left(\frac{-s_{c-2,c-1}}{-s_{c,c+1}}, \frac{-s_{c+2,c-2}}{-s_{c,c+1}} \right)}{s_{c,c+1}^3} \xrightarrow{c \parallel c+1} \frac{1}{2st} \frac{1}{s_{c,c+1}} \\
& + \frac{\ln^2 \left(\frac{-s}{-t} \right) + \pi^2}{2(s+t)^3} + \ln \left(\frac{-s_{c,c+1}}{-t} \right) \left[\frac{1}{t(s+t)^2} + \frac{1}{2t^2(s+t)} \right] \\
& + \ln \left(\frac{-s_{c,c+1}}{-s} \right) \left[\frac{1}{s(s+t)^2} + \frac{1}{2s^2(s+t)} \right] \\
& + \frac{1}{2t^2(s+t)} + \frac{1}{2s^2(s+t)} + \frac{1}{2st(s+t)} .
\end{aligned} \tag{III.13}$$

Appendix IV. Mixed Photon-Gluon Amplitudes

The same primitive amplitudes used to construct the two-quark $(n-2)$ -gluon amplitudes can also be used to construct amplitudes with one to $(n-2)$ photons replacing gluons. The two-quark $(n-2)$ -gluon color decompositions (2.3) and (2.4) are valid for external gauge bosons in $U(N_c)$ as well as $SU(N_c)$, because the Fierz subtraction term in the gluon propagator (eqn. (2.2)) does not contribute unless the gluon is sandwiched between two fundamental representation lines. Thus partial amplitudes with photons may be obtained by substituting the photon generator matrix (which is proportional to the identity matrix) into the appropriate color decomposition formula, and grouping together terms with the same color structure. We illustrate the construction explicitly for two-quark one-photon $(n-3)$ -gluon amplitudes.

At tree level, amplitudes with one photon have a color decomposition similar to that of pure nonabelian ones,

$$\mathcal{A}_n^{1\gamma \text{ tree}} = Q\sqrt{2}eg^{n-3} \sum_{\sigma \in S_{n-3}} (T^{a_{\sigma(3)}} \dots T^{a_{\sigma(n-1)}})_{i_1}{}^{\bar{i}_2} A_n^{1\gamma \text{ tree}}(1_{\bar{q}}, 2_q; \sigma(3), \dots, \sigma(n-1); n) , \tag{IV.1}$$

where e is the QED coupling constant, and Q is the charge of the quark. The photon is taken to be the last leg, n . The partial amplitude $A_n^{1\gamma \text{ tree}}$ is related by a ‘decoupling’ equation [6] to the pure nonabelian partial amplitudes A_n^{tree} ,

$$\begin{aligned}
A_n^{1\gamma \text{ tree}}(1_{\bar{q}}, 2_q; 3, \dots, n-1; n) &= A_n^{\text{tree}}(1_{\bar{q}}, 2_q; n, 3, 4, \dots, n-1) + A_n^{\text{tree}}(1_{\bar{q}}, 2_q; 3, n, 4, \dots, n-1) \\
&+ \dots + A_n^{\text{tree}}(1_{\bar{q}}, 2_q; 3, 4, \dots, n-1, n) .
\end{aligned} \tag{IV.2}$$

Equation (IV.2) is obtained by substituting the photon generator matrix $T^{a_n} \propto \mathbf{1}$ into eqn. (2.3) and collecting terms. Note that unwanted diagrams on the right-hand side of (IV.2) — those coupling the photon to gluon lines — cancel out in the sum.

We can perform similar decompositions at one loop. We keep only $\mathcal{O}(e)$ contributions; we do not include the $\mathcal{O}(e^3)$ contributions where an internal photon line is exchanged between two charged lines. If we set aside the n_f - and n_s -dependent pieces, then the color factors are again simply the color factors for the two-quark $(n-3)$ -gluon amplitude (2.5),

$$\mathcal{A}_n^{1\gamma}|_{n_f, n_s=0} = Q\sqrt{2}eg^{n-1} \sum_{j=1}^{n-2} \sum_{\sigma \in S_{n-3}/S_{n-1;j}} \text{Gr}_{n-1;j}^{(\bar{q}q)}(\sigma(3, \dots, n-1)) A_{n;j}^{1\gamma}(1_{\bar{q}}, 2_q; \sigma(3, \dots, n-1); n)|_{n_f, n_s=0}, \quad (\text{IV.3})$$

and the partial amplitudes with one photon are again given by sums over the two-quark rest-gluon partial amplitudes, inserting the photon in all inequivalent gluon locations,

$$\begin{aligned} & A_{n;j}^{1\gamma}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, \dots, n-1; n)|_{n_f, n_s=0} \\ &= A_{n;j+1}(1_{\bar{q}}, 2_q; n, 3, \dots, j+1; j+2, \dots, n-1) + \dots + A_{n;j+1}(1_{\bar{q}}, 2_q; 3, \dots, n, j+1; j+2, \dots, n-1) \\ & \quad + A_{n;j}(1_{\bar{q}}, 2_q; 3, \dots, j+1; n, j+2, \dots, n-1) + \dots + A_{n;j}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, \dots, n-1, n); \end{aligned} \quad (\text{IV.4})$$

the unwanted diagrams again cancel.

For those contributions that do contain a (charged) fermion or scalar loop, we must in effect consider separately the diagrams where the photon couples to the external fermion line, and the diagrams where it couples to the closed fermion (or scalar) loop, since these different fermion lines may have different charges. This can be accomplished (for $j > 1$) by considering separately the contributions where the photon replaces a gluon within the first set, $3 \dots j+1$, associated with the trace of gluon matrices in $\text{Gr}_{n-1;j}^{(\bar{q}q)}$, or within the second set, $j+2 \dots n$, associated with the string of gluon matrices whose i_2, \bar{i}_1 component appears in the color factor. In the former case, the factor of Q along with n_f (or n_s) will be replaced by the trace over the fermion charge matrix (respectively the scalar charge matrix); in the latter case, the amplitude will continue to appear with factors of Q and n_f (or n_s for scalar-loop contributions).

More concretely, define $A_{n;j}^{[1/2]}$ to be the coefficient of n_f/N_c in $A_{n;j}$,

$$A_{n;j}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \dots, n-1, n) = N_c \frac{\partial}{\partial n_f} A_{n;j}(1_{\bar{q}}, 2_q; 3, \dots, n-1, n). \quad (\text{IV.5})$$

Using equations (4.2), (4.4), we may relate $A_{n;j}^{[1/2]}$ to primitive amplitudes,

$$\begin{aligned} A_{n;1}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \dots, n-1, n) &= -A_n^f(1_{\bar{q}}, 2_q; 3, \dots, n-1, n) - A_n^s(1_{\bar{q}}, 2_q; 3, \dots, n-1, n), \\ A_{n;j>1}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \dots, n-1, n) &= (-1)^j \sum_{\sigma \in \text{COP}\{\alpha\}\{\beta\}} A_n^{R,[1/2]}(\sigma(1_{\bar{q}}, 2_q, 3, \dots, n)). \end{aligned} \quad (\text{IV.6})$$

We can then construct the two different parts of the one-photon partial amplitude for each $j > 1$,

as described above,

$$\begin{aligned}
& A_{n;j}^{1\gamma,[1/2]:I}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, \dots, n-1; n) \\
&= A_{n;j+1}^{[1/2]}(1_{\bar{q}}, 2_q; n, 3, \dots, j+1; j+2, \dots, n-1) + \dots + A_{n;j+1}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \dots, n, j+1; j+2, \dots, n-1), \\
& A_{n;j}^{1\gamma,[1/2]:II}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, \dots, n-1; n) \\
&= A_{n;j}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \dots, j+1; n, j+2, \dots, n-1) + \dots + A_{n;j}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \dots, j+1; j+2, \dots, n-1, n).
\end{aligned} \tag{IV.7}$$

For $j = 1$ the color flow is different, and we get instead

$$\begin{aligned}
A_{n;1}^{1\gamma,[1/2]:I}(1_{\bar{q}}, 2_q; 3, \dots, n-1; n) &= A_{n;1}^{[1/2]}(1_{\bar{q}}, 2_q; n, 3, \dots, n-1) + \dots + A_{n;1}^{[1/2]}(1_{\bar{q}}, 2_q; 3, \dots, n-1, n) \\
&\quad + A_n^{L,[1/2]}(1_{\bar{q}}, n, 2_q, 3, \dots, n-1), \\
A_{n;1}^{1\gamma,[1/2]:II}(1_{\bar{q}}, 2_q; 3, \dots, n-1; n) &= -A_n^{L,[1/2]}(1_{\bar{q}}, n, 2_q, 3, \dots, n-1).
\end{aligned} \tag{IV.8}$$

Using these pieces, we write out the n_f -dependent pieces in the full amplitude,

$$\begin{aligned}
\mathcal{A}_n^{1\gamma}|_{n_f} &= \frac{\text{Tr}_f(Q_f)}{N_c} \sqrt{2} e g^{n-1} \sum_{j=1}^{n-2} \sum_{\sigma \in S_{n-3}/S_{n-1;j}} \text{Gr}_{n-1;j}^{(\bar{q}q)}(\sigma(3, \dots, n-1)) A_{n;j}^{1\gamma,[1/2]:I}(1_{\bar{q}}, 2_q; \sigma(3, \dots, n-1); n) \\
&\quad + \frac{Q_{n_f}}{N_c} \sqrt{2} e g^{n-1} \sum_{j=1}^{n-2} \sum_{\sigma \in S_{n-3}/S_{n-1;j}} \text{Gr}_{n-1;j}^{(\bar{q}q)}(\sigma(3, \dots, n-1)) A_{n;j}^{1\gamma,[1/2]:II}(1_{\bar{q}}, 2_q; \sigma(3, \dots, n-1); n),
\end{aligned} \tag{IV.9}$$

where Q_f is the fermion charge matrix, and Tr_f represents the trace over flavors. An analogous decomposition holds for the contributions proportional to n_s .

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